

# Dirichlet's function is not integrable; Thomae's is

Lesson 81 · Module 7 — The Riemann Integral · Real Analysis · [dailymaths.uk/analysis](http://dailymaths.uk/analysis)

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## Practice problems

1. Let  $f(x)$  be the Dirichlet function on  $[0, 1]$ . Prove that for any partition  $P$ , the lower Darboux sum  $L(f, P)$  is exactly 0.
2. Consider a function  $g(x)$  on  $[0, 1]$  that is 0 everywhere except at  $x = 1/3$  and  $x = 2/3$ , where  $g(x) = 1$ . Use the integrability criterion to show  $g$  is integrable.
3. If a function  $f$  is Riemann integrable on  $[a, b]$ , and we change its value at a single point  $x_0$  to a new value  $y$ , is the new function  $f^*$  still integrable? Justify your answer.

## Solutions

1. Let  $f(x)$  be the Dirichlet function on  $[0, 1]$ . Prove that for any partition  $P$ , the lower Darboux sum  $L(f, P)$  is exactly 0.
  - Recall that  $L(f, P) = \sum m_k \Delta x_k$ , where  $m_k = \inf_{x \in [x_{k-1}, x_k]} f(x)$ .
  - In any sub-interval  $[x_{k-1}, x_k]$ , there exists at least one irrational number  $x$  because the irrationals are dense in  $\mathbb{R}$ .
  - For any irrational  $x$ ,  $f(x) = 0$ .
  - Since  $f(x) \geq 0$  for all  $x$ , the infimum  $m_k$  must be 0.
  - Therefore,  $L(f, P) = \sum 0 \cdot \Delta x_k = 0$ .

**Answer:**  $L(f, P) = 0$

2. Consider a function  $g(x)$  on  $[0, 1]$  that is 0 everywhere except at  $x = 1/3$  and  $x = 2/3$ , where  $g(x) = 1$ . Use the integrability criterion to show  $g$  is integrable.
  - The lower sum  $L(g, P)$  is always 0 because every interval contains points where  $g(x) = 0$ .
  - For any  $\epsilon > 0$ , choose a partition that isolates  $1/3$  and  $2/3$  in intervals of width  $\epsilon/4$ .
  - The upper sum  $U(g, P)$  consists of two intervals where  $M_k = 1$  and the rest where  $M_k = 0$ .
  - Thus  $U(g, P) = 1 \cdot (\epsilon/4) + 1 \cdot (\epsilon/4) + 0 = \epsilon/2$ .
  - Since  $U(g, P) - L(g, P) = \epsilon/2 < \epsilon$ , the function is integrable.

**Answer:**  $\int_0^1 g(x) \, dx = 0$

3. If a function  $f$  is Riemann integrable on  $[a, b]$ , and we change its value at a single point  $x_0$  to a new value  $y$ , is the new function  $f^*$  still integrable? Justify your answer.
  - Let  $P$  be a partition for  $f$  such that  $U(f, P) - L(f, P) < \epsilon/2$ .
  - Isolate  $x_0$  in a sub-interval  $I_k$  of width  $\delta$ .
  - The change in the upper sum is at most  $|y - f(x_0)| \cdot \delta$ .
  - The change in the lower sum is at most  $|y - f(x_0)| \cdot \delta$ .
  - By choosing  $\delta$  small enough, the total change in  $U - L$  can be made less than  $\epsilon/2$ .
  - Thus  $U(f^*, P) - L(f^*, P) < \epsilon$ , so  $f^*$  is integrable.

**Answer:** Yes, it is still integrable.