Which functions are integrable? Continuous and monotone functions

Practice problems

1. Determine if the function $f(x) = \begin{cases} x^2 & 0 \le x < 1 \\ 2 & x = 1 \end{cases}$ is Riemann integrable on $[0, 1]$. Justify your answer using the concepts from today.
2. Let f be a continuous function on $[a, b]$. If we use a uniform partition with n sub-intervals, and we know that for a given $\epsilon > 0$, the oscillation $M_k - m_k < \epsilon$ for all k, show that $U(f, P) - L(f, P) < \epsilon(b - a)$.
3. Is every integrable function on $[0, 1]$ necessarily monotone or continuous? Provide a counterexample or a brief explanation.

Solutions

1. Determine if the function $f(x) = \begin{cases} x^2 & 0 \leq x < 1 \\ 2 & x = 1 \end{cases}$ is Riemann integrable on $[0, 1]$. Justify your answer using the concepts from today.

   - Check for continuity: The function has a jump discontinuity at $x = 1$ because $\lim_{x \to 1^-} f(x) = 1$ but $f(1) = 2$. Thus, it is not continuous.
   - Check for monotonicity: For $0 \leq x < 1$, $f(x) = x^2$ is strictly increasing. At $x = 1$, $f(1) = 2$, which is greater than all previous values. Thus, f is non-decreasing on $[0, 1]$.
   - Apply the theorem: Since f is bounded and monotone on the closed interval $[0, 1]$, it is Riemann integrable.

   Answer: Yes, it is integrable because it is monotone.

2. Let f be a continuous function on $[a, b]$. If we use a uniform partition with n sub-intervals, and we know that for a given $\epsilon > 0$, the oscillation $M_k - m_k < \epsilon$ for all k, show that $U(f, P) - L(f, P) < \epsilon(b - a)$.

   - The difference is $U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k$.
   - Substitute the bound $M_k - m_k < \epsilon$: the sum is less than $\sum_{k=1}^{n} \epsilon \Delta x_k$.
   - Factor out ϵ: $\epsilon \sum_{k=1}^{n} \Delta x_k$.
   - The sum of the widths of the sub-intervals is the total length $b - a$.
   - Therefore, the difference is less than $\epsilon(b - a)$.

   Answer: The difference is bounded by $\epsilon(b - a)$.

3. Is every integrable function on $[0, 1]$ necessarily monotone or continuous? Provide a counterexample or a brief explanation.

   - Consider a function with a single jump discontinuity that is not monotone, such as $f(x) = 0$ for $x \in [0, 1/3)$, $f(x) = 1$ for $x \in [1/3, 2/3)$, and $f(x) = 0$ for $x \in [2/3, 1]$.
   - This function is not continuous because of the jumps at $1/3$ and $2/3$.
   - This function is not monotone because it increases and then decreases.
   - However, it is a step function with finitely many jumps, which is always Riemann integrable.
   - Thus, continuity and monotonicity are sufficient but not necessary.

   Answer: No, a function can be integrable without being monotone or continuous (e.g., a step function with multiple jumps).