

Integrating x^2 from the definition — once, honestly

Lesson 79 · Module 7 — The Riemann Integral · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Use the Riemann definition (with a regular partition of n sub-intervals) to compute $\int_0^1 x \, dx$.
2. For $f(x) = x^2$ on $[0, 1]$, find the explicit expression for the difference $U(f, P_n) - L(f, P_n)$ for a regular partition of n intervals.
3. Using the Riemann definition and the formula $\sum_{k=1}^n k^3 = [n(n+1)/2]^2$, compute $\int_0^1 x^3 \, dx$.

Solutions

1. Use the Riemann definition (with a regular partition of n sub-intervals) to compute $\int_0^1 x \, dx$.
 - Set $\Delta x = 1/n$ and $x_k = k/n$.
 - The upper sum is $U(f, P_n) = \sum_{k=1}^n (k/n)(1/n) = (1/n^2) \sum k$.
 - Using $\sum_{k=1}^n k = n(n+1)/2$, we get $U(f, P_n) = n(n+1)/(2n^2) = (n^2 + n)/(2n^2)$.
 - Taking the limit as $n \rightarrow \infty$, we get $1/2$.

Answer: $1/2$

2. For $f(x) = x^2$ on $[0, 1]$, find the explicit expression for the difference $U(f, P_n) - L(f, P_n)$ for a regular partition of n intervals.
 - From the lesson, $U(f, P_n) = (2n^3 + 3n^2 + n)/(6n^3)$.
 - From the lesson, $L(f, P_n) = (2n^3 - 3n^2 + n)/(6n^3)$.
 - Subtracting these gives $(6n^2)/(6n^3) = 1/n$.

Answer: $1/n$

3. Using the Riemann definition and the formula $\sum_{k=1}^n k^3 = [n(n+1)/2]^2$, compute $\int_0^1 x^3 \, dx$.
 - Set $\Delta x = 1/n$ and $x_k = k/n$.
 - The upper sum is $U(f, P_n) = \sum_{k=1}^n (k/n)^3 (1/n) = (1/n^4) \sum k^3$.
 - Substitute the formula: $U(f, P_n) = (1/n^4)[n^2(n+1)^2/4] = (n+1)^2/(4n^2)$.
 - Taking the limit as $n \rightarrow \infty$, $(n^2 + 2n + 1)/(4n^2)$ approaches $1/4$.

Answer: $1/4$