

The Riemann Integral Defined; the Integrability Criterion

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Practice problems

1. Use the integrability criterion to prove that the function $f(x) = 2$ for $0 \leq x < 1$ and $f(x) = 5$ for $1 \leq x \leq 2$ is Riemann integrable on $[0, 2]$.
2. Prove that the function $f(x) = x^2$ on $[0, 1]$ is Riemann integrable using the criterion. Hint: use a uniform partition into n intervals.
3. Let f be a bounded function on $[a, b]$. If f is integrable, prove that for any partition P , $L(f, P) \leq \int_a^b f \leq U(f, P)$.

Solutions

1. Use the integrability criterion to prove that the function $f(x) = 2$ for $0 \leq x < 1$ and $f(x) = 5$ for $1 \leq x \leq 2$ is Riemann integrable on $[0, 2]$.

- Choose a partition P that includes the point $x = 1$.
- The sub-intervals are $[0, 1]$ and $[1, 2]$.
- On $[0, 1]$, $m_1 = 2$ and $M_1 = 2$. On $[1, 2]$, $m_2 = 5$ and $M_2 = 5$.
- The lower sum $L(f, P) = 2(1) + 5(1) = 7$.
- The upper sum $U(f, P) = 2(1) + 5(1) = 7$.
- Since $U(f, P) - L(f, P) = 0 < \epsilon$ for any $\epsilon > 0$, the function is integrable.

Answer: The function is integrable with $\int_0^2 f(x) dx = 7$.

2. Prove that the function $f(x) = x^2$ on $[0, 1]$ is Riemann integrable using the criterion. Hint: use a uniform partition into n intervals.

- For a uniform partition, $\Delta x = 1/n$.
- On the k -th interval $[(k-1)/n, k/n]$, $m_k = ((k-1)/n)^2$ and $M_k = (k/n)^2$.
- The difference $M_k - m_k = (k^2 - (k-1)^2)/n^2 = (2k-1)/n^2$.
- The total gap $U - L = \sum_{k=1}^n (2k-1)/n^2 \cdot (1/n) = (1/n^3) \sum (2k-1)$.
- The sum of the first n odd numbers is n^2 , so $U - L = n^2/n^3 = 1/n$.
- For any $\epsilon > 0$, choose $n > 1/\epsilon$ to satisfy the criterion.

Answer: The function is integrable because $U - L = 1/n \rightarrow 0$ as $n \rightarrow \infty$.

3. Let f be a bounded function on $[a, b]$. If f is integrable, prove that for any partition P , $L(f, P) \leq \int_a^b f \leq U(f, P)$.

- By definition, the lower integral is the supremum of all lower sums: $\underline{\int} f = \sup_P L(f, P)$.
- By the property of the supremum, any specific lower sum $L(f, P)$ must be less than or equal to the supremum.
- Similarly, the upper integral is the infimum of all upper sums: $\overline{\int} f = \inf_P U(f, P)$.
- Any specific upper sum $U(f, P)$ must be greater than or equal to the infimum.
- Since the function is integrable, $\int f = \underline{\int} f = \overline{\int} f$.
- Therefore, $L(f, P) \leq \int f \leq U(f, P)$.

Answer: The result follows directly from the definitions of supremum and infimum.