

What should 'area' mean? Darboux sums, upper and lower

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Practice problems

1. Let $f(x) = 1$ for $0 \leq x < 1$ and $f(x) = 2$ for $1 \leq x \leq 2$. For the partition $P = \{0, 0.5, 1.5, 2\}$, compute the lower sum $L(f, P)$ and the upper sum $U(f, P)$.
2. For $f(x) = x^2$ on $[0, 1]$ with a partition of n equal sub intervals, find the expression for the upper sum $U(f, P)$ and its limit as $n \rightarrow \infty$.
3. Prove that if f is a constant function $f(x) = C$ on $[a, b]$, then it is Riemann integrable and its integral is $C(b - a)$.

Solutions

1. Let $f(x) = 1$ for $0 \leq x < 1$ and $f(x) = 2$ for $1 \leq x \leq 2$. For the partition $P = \{0, 0.5, 1.5, 2\}$, compute the lower sum $L(f, P)$ and the upper sum $U(f, P)$.

- Interval 1 $[0, 0.5]$: $m_1 = 1, M_1 = 1, \Delta x_1 = 0.5$.
- Interval 2 $[0.5, 1.5]$: $m_2 = 1, M_2 = 2, \Delta x_2 = 1.0$.
- Interval 3 $[1.5, 2]$: $m_3 = 2, M_3 = 2, \Delta x_3 = 0.5$.
- Lower sum: $1(0.5) + 1(1.0) + 2(0.5) = 0.5 + 1.0 + 1.0 = 2.5$.
- Upper sum: $1(0.5) + 2(1.0) + 2(0.5) = 0.5 + 2.0 + 1.0 = 3.5$.

Answer: $L(f, P) = 2.5, U(f, P) = 3.5$

2. For $f(x) = x^2$ on $[0, 1]$ with a partition of n equal sub intervals, find the expression for the upper sum $U(f, P)$ and its limit as $n \rightarrow \infty$.

- Width $\Delta x = 1/n$.
- On $[(k-1)/n, k/n]$, the supremum is $M_k = (k/n)^2$.
- Upper sum $U(f, P) = \sum_{k=1}^n (k/n)^2 (1/n) = \frac{1}{n^3} \sum_{k=1}^n k^2$.
- Using the formula $\sum k^2 = \frac{n(n+1)(2n+1)}{6}$, we get $U(f, P) = \frac{n(n+1)(2n+1)}{6n^3}$.
- Taking the limit as $n \rightarrow \infty$, the leading terms are $2n^3/6n^3 = 1/3$.

Answer: $U(f, P) = \frac{(n+1)(2n+1)}{6n^2}$, \text{ limit is } 1/3

3. Prove that if f is a constant function $f(x) = C$ on $[a, b]$, then it is Riemann integrable and its integral is $C(b-a)$.

- For any partition P , on every sub interval $[x_{k-1}, x_k]$, the infimum $m_k = C$ and the supremum $M_k = C$.
- The lower sum $L(f, P) = \sum_{k=1}^n C(x_k - x_{k-1}) = C \sum (x_k - x_{k-1}) = C(b-a)$.
- The upper sum $U(f, P) = \sum_{k=1}^n C(x_k - x_{k-1}) = C(b-a)$.
- Since $L(f, P) = U(f, P) = C(b-a)$ for all P , the supremum of lower sums and infimum of upper sums are both $C(b-a)$.
- Thus, the function is integrable with integral $C(b-a)$.

Answer: $\int_a^b C \, dx = C(b-a)$