

Module 6 Review and Self-Test

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Practice problems

1. Use the Mean Value Theorem to prove that $e^x \geq 1 + x$ for all $x \geq 0$.
2. Let $f(x) = x^3$ for $x < 0$ and $f(x) = x^2$ for $x \geq 0$. Is f differentiable at $x = 0$? Justify your answer.
3. Suppose f is differentiable on $[0, 1]$ and $f'(x) \leq 2$ for all $x \in (0, 1)$. If $f(0) = 3$, what is the maximum possible value of $f(1)$?

Solutions

1. Use the Mean Value Theorem to prove that $e^x \geq 1 + x$ for all $x \geq 0$.

- Define $f(t) = e^t$ on the interval $[0, x]$.
- The function is continuous on $[0, x]$ and differentiable on $(0, x)$.
- By the MVT, there exists $c \in (0, x)$ such that $f'(c) = \frac{e^x - e^0}{x - 0}$.
- Since $f'(t) = e^t$, we have $e^c = \frac{e^x - 1}{x}$.
- Because $c > 0$, we know $e^c > e^0 = 1$.
- Thus, $\frac{e^x - 1}{x} > 1$, which implies $e^x - 1 > x$, or $e^x > 1 + x$.

Answer: $e^x \geq 1 + x$

2. Let $f(x) = x^3$ for $x < 0$ and $f(x) = x^2$ for $x \geq 0$. Is f differentiable at $x = 0$? Justify your answer.

- Check the left-hand derivative: $\lim_{h \rightarrow 0^-} \frac{h^3 - 0}{h} = \lim_{h \rightarrow 0^-} h^2 = 0$.
- Check the right-hand derivative: $\lim_{h \rightarrow 0^+} \frac{h^2 - 0}{h} = \lim_{h \rightarrow 0^+} h = 0$.
- Since both one-sided limits exist and are equal to 0, the derivative exists at $x = 0$ and $f'(0) = 0$.

Answer: Yes, f is differentiable at $x = 0$ with $f'(0) = 0$.

3. Suppose f is differentiable on $[0, 1]$ and $f'(x) \leq 2$ for all $x \in (0, 1)$. If $f(0) = 3$, what is the maximum possible value of $f(1)$?

- Apply the Mean Value Theorem to f on $[0, 1]$.
- There exists $c \in (0, 1)$ such that $f'(c) = \frac{f(1) - f(0)}{1 - 0}$.
- Substitute the known values: $f'(c) = f(1) - 3$.
- We are given that $f'(x) \leq 2$ for all x , so $f'(c) \leq 2$.
- Therefore, $f(1) - 3 \leq 2$, which implies $f(1) \leq 5$.

Answer: 5