

L'Hopital's rule — proved at last, with its fine print

Lesson 75 · Module 6 — Differentiation, Rigorously · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Evaluate the limit $\lim_{x \rightarrow 0} \frac{x - \sin(x)}{x^3}$.
2. Determine if L'Hopital's rule can be used to find $\lim_{x \rightarrow \infty} \frac{x + \sin(x)}{x}$.
3. Evaluate $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x}$.

Solutions

1. Evaluate the limit $\lim_{x \rightarrow 0} \frac{x - \sin(x)}{x^3}$.

- Check form: $0 - \sin(0) = 0$ and $0^3 = 0$. It is $0/0$.
- Apply L'Hopital: $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{3x^2}$. This is still $0/0$.
- Apply L'Hopital again: $\lim_{x \rightarrow 0} \frac{\sin(x)}{6x}$. This is still $0/0$.
- Apply L'Hopital a third time: $\lim_{x \rightarrow 0} \frac{\cos(x)}{6} = \frac{1}{6}$.

Answer: $1/6$

2. Determine if L'Hopital's rule can be used to find $\lim_{x \rightarrow \infty} \frac{x + \sin(x)}{x}$.

- Check form: $x + \sin(x) \rightarrow \infty$ and $x \rightarrow \infty$. It is ∞/∞ .
- Take derivatives: $f'(x) = 1 + \cos(x)$ and $g'(x) = 1$.
- Examine the limit of the ratio: $\lim_{x \rightarrow \infty} (1 + \cos(x))$.
- This limit does not exist because $\cos(x)$ oscillates between -1 and 1 .
- Since the limit of the derivatives does not exist, L'Hopital's rule is inconclusive.

Answer: No, the rule is inconclusive because the limit of the derivatives does not exist.

3. Evaluate $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x}$.

- Check form: $\ln(x) \rightarrow -\infty$ and $1/x \rightarrow \infty$. This is ∞/∞ .
- Take derivatives: $f'(x) = 1/x$ and $g'(x) = -1/x^2$.
- Form the ratio: $\frac{1/x}{-1/x^2} = \frac{1}{x} \cdot (-x^2) = -x$.
- Evaluate the limit: $\lim_{x \rightarrow 0^+} (-x) = 0$.

Answer: 0