

Darboux's Theorem

Lesson 74 · Module 6 — Differentiation, Rigorously · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Determine if the function $g(x) = \begin{cases} x & x < 0 \\ x + 1 & x \geq 0 \end{cases}$ can be the derivative of some differentiable function f on the interval $[-1, 1]$. Justify your answer using Darboux's Theorem.
2. Suppose f is differentiable on $[0, 2]$ with $f'(0) = 1$ and $f'(2) = 4$. Prove that there exists at least one $c \in (0, 2)$ such that $f'(c) = 3$.
3. Let f be differentiable on $[a, b]$. If $f'(x) \neq 0$ for all $x \in [a, b]$, prove that f is strictly monotonic on $[a, b]$.

Solutions

1. Determine if the function $g(x) = \begin{cases} x & x < 0 \\ x + 1 & x \geq 0 \end{cases}$ can be the derivative of some differentiable function f on the interval $[-1, 1]$. Justify your answer using Darboux's Theorem.
- Check the values of $g(x)$ at the endpoints: $g(-1) = -1$ and $g(1) = 2$.
 - According to Darboux's Theorem, if g is a derivative, it must take every value between -1 and 2 .
 - Observe that $g(x)$ never takes the value 0.5 , because for $x < 0$, $g(x) < 0$, and for $x \geq 0$, $g(x) \geq 1$.
 - Since g skips values in the interval $(-1, 2)$, it cannot be a derivative.

Answer: No, it cannot be a derivative because it fails the Intermediate Value Property.

2. Suppose f is differentiable on $[0, 2]$ with $f'(0) = 1$ and $f'(2) = 4$. Prove that there exists at least one $c \in (0, 2)$ such that $f'(c) = 3$.
- Identify the hypotheses: f is differentiable on $[0, 2]$.
 - Identify the endpoint values of the derivative: $f'(0) = 1$ and $f'(2) = 4$.
 - Note that the target value $y = 3$ satisfies $1 < 3 < 4$.
 - By Darboux's Theorem, since $f'(0) < 3 < f'(2)$, there must exist $c \in (0, 2)$ such that $f'(c) = 3$.

Answer: The existence of c is guaranteed by Darboux's Theorem.

3. Let f be differentiable on $[a, b]$. If $f'(x) \neq 0$ for all $x \in [a, b]$, prove that f is strictly monotonic on $[a, b]$.
- Assume for contradiction that f' is not strictly monotonic. This would mean f' takes both positive and negative values.
 - If there exist x_1, x_2 such that $f'(x_1) < 0$ and $f'(x_2) > 0$, then by Darboux's Theorem, there must exist some c between x_1 and x_2 such that $f'(c) = 0$.
 - However, the hypothesis states $f'(x) \neq 0$ for all x .
 - Therefore, f' must be either always positive or always negative.
 - From Lesson 73, a derivative that is strictly positive (or negative) implies the function is strictly increasing (or decreasing).

Answer: Since f' cannot change sign without passing through zero, it must be constant in sign, implying strict monotonicity.