

MVT consequences: monotonicity, derivative tests — all justified

Lesson 73 · Module 6 — Differentiation, Rigorously · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Suppose f is differentiable on $[a, b]$ and $|f'(x)| \leq k$ for all $x \in (a, b)$. Prove that $|f(b) - f(a)| \leq k|b - a|$.
2. Let $f(x) = x^3 - 3x$. Find the intervals of monotonicity and identify the local extrema.
3. Prove that if $f'(x) \geq 0$ for all x in an interval, then f is non-decreasing on that interval.

Solutions

1. Suppose f is differentiable on $[a, b]$ and $|f'(x)| \leq k$ for all $x \in (a, b)$. Prove that $|f(b) - f(a)| \leq k|b - a|$.
- By the Mean Value Theorem, there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.
 - Take the absolute value of both sides: $|f(b) - f(a)| = |f'(c)||b - a|$.
 - Since $|f'(x)| \leq k$ for all x , we know $|f'(c)| \leq k$.
 - Therefore, $|f(b) - f(a)| \leq k|b - a|$.

Answer: $|f(b) - f(a)| \leq k|b - a|$

2. Let $f(x) = x^3 - 3x$. Find the intervals of monotonicity and identify the local extrema.
- Compute the derivative: $f'(x) = 3x^2 - 3$.
 - Set $f'(x) = 0$ to find critical points: $3(x^2 - 1) = 0 \implies x = \pm 1$.
 - Test intervals: For $x < -1$, $f'(x) > 0$ (increasing). For $-1 < x < 1$, $f'(x) < 0$ (decreasing). For $x > 1$, $f'(x) > 0$ (increasing).
 - At $x = -1$, sign changes $(+) \rightarrow (-)$, so local max. At $x = 1$, sign changes $(-) \rightarrow (+)$, so local min.

Answer: Increasing on $(-\infty, -1) \cup (1, \infty)$, decreasing on $(-1, 1)$; local max at $x = -1$, local min at $x = 1$.

3. Prove that if $f'(x) \geq 0$ for all x in an interval, then f is non-decreasing on that interval.
- Pick any x_1, x_2 in the interval such that $x_1 < x_2$.
 - By the Mean Value Theorem, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some $c \in (x_1, x_2)$.
 - We are given $f'(c) \geq 0$ and we know $x_2 - x_1 > 0$.
 - The product of two non-negative numbers is non-negative, so $f(x_2) - f(x_1) \geq 0$.
 - Thus $f(x_2) \geq f(x_1)$, which is the definition of a non-decreasing function.

Answer: $f(x_2) \geq f(x_1)$