

The Mean Value Theorem — proved

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Practice problems

1. Find all values of c that satisfy the Mean Value Theorem for $f(x) = \sqrt{x}$ on the interval $[1, 9]$.
2. Let f be a differentiable function such that $f'(x) \leq 3$ for all x . If $f(0) = 2$, what is the maximum possible value of $f(4)$?
3. Prove that if $f'(x) = 0$ for all x in an interval (a, b) , then f is constant on that interval.

Solutions

1. Find all values of c that satisfy the Mean Value Theorem for $f(x) = \sqrt{x}$ on the interval $[1, 9]$.

- Calculate the average slope: $\frac{f(9)-f(1)}{9-1} = \frac{3-1}{8} = \frac{2}{8} = \frac{1}{4}$.
- Find the derivative: $f'(x) = \frac{1}{2\sqrt{x}}$.
- Set the derivative equal to the average slope: $\frac{1}{2\sqrt{c}} = \frac{1}{4}$.
- Solve for c : $2\sqrt{c} = 4 \implies \sqrt{c} = 2 \implies c = 4$.
- Check if c is in the interval $(1, 9)$: 4 is indeed in the interval.

Answer: $c = 4$

2. Let f be a differentiable function such that $f'(x) \leq 3$ for all x . If $f(0) = 2$, what is the maximum possible value of $f(4)$?

- Apply the MVT on the interval $[0, 4]$: $\frac{f(4)-f(0)}{4-0} = f'(c)$ for some $c \in (0, 4)$.
- Substitute known values: $\frac{f(4)-2}{4} = f'(c)$.
- Use the given bound $f'(x) \leq 3$: $\frac{f(4)-2}{4} \leq 3$.
- Solve for $f(4)$: $f(4) - 2 \leq 12 \implies f(4) \leq 14$.

Answer: 14

3. Prove that if $f'(x) = 0$ for all x in an interval (a, b) , then f is constant on that interval.

- Pick any two points x_1, x_2 in the interval such that $x_1 < x_2$.
- Apply the MVT to f on the closed interval $[x_1, x_2]$.
- There exists $c \in (x_1, x_2)$ such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$.
- Since $f'(x) = 0$ for all x , we have $f'(c) = 0$.
- Thus, $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$, which implies $f(x_2) - f(x_1) = 0$, so $f(x_1) = f(x_2)$.
- Since this holds for any x_1, x_2 , the function must be constant.

Answer: $f(x) = C$