Local extrema and Fermat's theorem; Rolle — proved

Practice problems

1. Let $f(x) = x^3 - 3x^2 + 2x$ on the interval $[0, 1]$. Show that the hypotheses of Rolle's Theorem are satisfied and find all points $c \in (0, 1)$ where $f'(c) = 0$.

2. Consider $f(x) = x^{2/3}$ on the interval $[-1, 1]$. Does Rolle's Theorem apply? If not, which hypothesis is violated?

3. Suppose f is differentiable on (a, b) and $f'(x) \neq 0$ for all $x \in (a, b)$. Prove that $f(a)$ cannot equal $f(b)$ if f is continuous on $[a, b]$.

Solutions

1. Let $f(x) = x^3 - 3x^2 + 2x$ on the interval $[0, 1]$. Show that the hypotheses of Rolle's Theorem are satisfied and find all points $c \in (0, 1)$ where $f'(c) = 0$.

   - Check endpoints: $f(0) = 0$ and $f(1) = 1 - 3 + 2 = 0$.
   - Continuity and differentiability: f is a polynomial, so it is continuous on $[0, 1]$ and differentiable on $(0, 1)$.
   - Compute derivative: $f'(x) = 3x^2 - 6x + 2$.
   - Solve $3x^2 - 6x + 2 = 0$ using the quadratic formula: $x = \frac{6 \pm \sqrt{36-24}}{6} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{3}}{3}$.
   - Check interval: $1 + \frac{\sqrt{3}}{3} > 1$, but $1 - \frac{\sqrt{3}}{3} \approx 0.422$, which is in $(0, 1)$.

   Answer: $c = 1 - \frac{\sqrt{3}}{3}$

2. Consider $f(x) = x^{2/3}$ on the interval $[-1, 1]$. Does Rolle's Theorem apply? If not, which hypothesis is violated?

   - Check endpoints: $f(-1) = (-1)^{2/3} = 1$ and $f(1) = 1^{2/3} = 1$. Endpoints are equal.
   - Check continuity: $f(x)$ is continuous on $[-1, 1]$.
   - Check differentiability: $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$.
   - At $x = 0$, the derivative is undefined. Since $0 \in (-1, 1)$, the function is not differentiable on the open interval.

   Answer: No, Rolle's Theorem does not apply because f is not differentiable at $x = 0$.

3. Suppose f is differentiable on (a, b) and $f'(x) \neq 0$ for all $x \in (a, b)$. Prove that $f(a)$ cannot equal $f(b)$ if f is continuous on $[a, b]$.

   - Assume for contradiction that $f(a) = f(b)$.
   - Since f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, all hypotheses of Rolle's Theorem are satisfied.
   - By Rolle's Theorem, there must exist some $c \in (a, b)$ such that $f'(c) = 0$.
   - However, the problem states that $f'(x) \neq 0$ for all $x \in (a, b)$.
   - This is a contradiction. Therefore, the assumption $f(a) = f(b)$ must be false.

   Answer: Proven by contradiction using Rolle's Theorem.