

The Weierstrass monster

Lesson 70 · Module 6 — Differentiation, Rigorously · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Show that the function $g(x) = \sum_{n=1}^{\infty} \frac{\sin(2^n x)}{3^n}$ is continuous for all x using the comparison test with a geometric series.
2. For the Weierstrass function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$, if $a = 0.4$ and $b = 5$, does the condition $ab > 1 + 1.5\pi$ hold?
3. Explain why the term-by-term derivative of $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ fails to converge when $ab > 1$.

Solutions

1. Show that the function $g(x) = \sum_{n=1}^{\infty} \frac{\sin(2^n x)}{3^n}$ is continuous for all x using the comparison test with a geometric series.

- Observe that $|\sin(2^n x)| \leq 1$ for all n and x .
- Thus, the absolute value of the n -th term is $|\frac{\sin(2^n x)}{3^n}| \leq \frac{1}{3^n}$.
- The series $\sum_{n=1}^{\infty} (1/3)^n$ is a geometric series with $r = 1/3 < 1$, so it converges.
- By the comparison test for series, the series for $g(x)$ converges absolutely for all x , and since the terms are continuous, the sum is continuous.

Answer: The series is bounded by the convergent geometric series $\sum (1/3)^n$, ensuring continuity.

2. For the Weierstrass function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$, if $a = 0.4$ and $b = 5$, does the condition $ab > 1 + 1.5\pi$ hold?

- Calculate the product ab : $0.4 \times 5 = 2.0$.
- Calculate the threshold: $1 + 1.5\pi \approx 1 + 1.5(3.14159) \approx 1 + 4.712 = 5.712$.
- Compare the two values: $2.0 < 5.712$.

Answer: No, the condition does not hold.

3. Explain why the term-by-term derivative of $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ fails to converge when $ab > 1$.

- The n -th term of the derivative series is $f'_n(x) = -\pi a^n b^n \sin(b^n \pi x)$.
- The amplitude of this term is $|\pi(ab)^n|$.
- If $ab > 1$, then $(ab)^n \rightarrow \infty$ as $n \rightarrow \infty$.
- Since the amplitudes of the terms grow without bound, the series of derivatives cannot converge to a finite value.

Answer: The amplitudes of the derivative terms grow exponentially as $(ab)^n$.