

$x^2 \sin(1/x)$ and friends

Lesson 69 · Module 6 — Differentiation, Rigorously · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Let $f(x) = x^{3/2} \sin(1/x)$ for $x > 0$ and $f(0) = 0$. Is f differentiable at $x = 0$? If so, is f' continuous at $x = 0$?
2. For $f(x) = x^n \sin(1/x)$ with $f(0) = 0$ and $n > 0$, find the smallest integer n such that f' is continuous at $x = 0$.
3. Let $f(x) = x^3 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Compute the value of $f'(1/\pi)$.

Solutions

1. Let $f(x) = x^{3/2} \sin(1/x)$ for $x > 0$ and $f(0) = 0$. Is f differentiable at $x = 0$? If so, is f' continuous at $x = 0$?

- Check differentiability at 0: $f'(0) = \lim_{x \rightarrow 0^+} \frac{x^{3/2} \sin(1/x) - 0}{x} = \lim_{x \rightarrow 0^+} x^{1/2} \sin(1/x)$.
- Since $|x^{1/2} \sin(1/x)| \leq x^{1/2}$ and $x^{1/2} \rightarrow 0$, the limit is 0. So $f'(0) = 0$.
- Compute $f'(x)$ for $x > 0$: $f'(x) = \frac{3}{2}x^{1/2} \sin(1/x) + x^{3/2} \cos(1/x) \cdot (-1/x^2) = \frac{3}{2}x^{1/2} \sin(1/x) - x^{-1/2} \cos(1/x)$.
- Analyze $\lim_{x \rightarrow 0^+} f'(x)$: The first term goes to 0, but the second term $x^{-1/2} \cos(1/x)$ is unbounded and oscillates as $x \rightarrow 0$.
- Therefore, the limit does not exist, and f' is not continuous at 0.

Answer: Differentiable at $x = 0$ with $f'(0) = 0$, but f' is not continuous at $x = 0$.

2. For $f(x) = x^n \sin(1/x)$ with $f(0) = 0$ and $n > 0$, find the smallest integer n such that f' is continuous at $x = 0$.

- For f to be differentiable at 0, we need $\lim_{x \rightarrow 0} x^{n-1} \sin(1/x) = 0$, which requires $n > 1$.
- The derivative for $x \neq 0$ is $f'(x) = nx^{n-1} \sin(1/x) - x^{n-2} \cos(1/x)$.
- For f' to be continuous at 0, we need $\lim_{x \rightarrow 0} f'(x) = f'(0) = 0$.
- The term $nx^{n-1} \sin(1/x)$ goes to 0 if $n > 1$.
- The term $x^{n-2} \cos(1/x)$ goes to 0 if $n - 2 > 0$, which means $n > 2$.
- The smallest integer n satisfying $n > 2$ is $n = 3$.

Answer: $n = 3$

3. Let $f(x) = x^3 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Compute the value of $f'(1/\pi)$.

- Use the derivative formula for $x \neq 0$: $f'(x) = 3x^2 \sin(1/x) - x \cos(1/x)$.
- Substitute $x = 1/\pi$: $f'(1/\pi) = 3(1/\pi)^2 \sin(\pi) - (1/\pi) \cos(\pi)$.
- Since $\sin(\pi) = 0$ and $\cos(\pi) = -1$, we have $f'(1/\pi) = 3(1/\pi)^2(0) - (1/\pi)(-1)$.
- This simplifies to $0 + 1/\pi = 1/\pi$.

Answer: $1/\pi$