

The derivative as a limit — differentiability implies continuity

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Practice problems

1. Use the limit definition of the derivative to find $f'(a)$ for $f(x) = \frac{1}{x}$ where $a \neq 0$.
2. Let $f(x) = x^{1/3}$. Determine if f is continuous at $x = 0$ and if it is differentiable at $x = 0$.
3. Prove that if f is not continuous at a , then it cannot be differentiable at a .

Solutions

1. Use the limit definition of the derivative to find $f'(a)$ for $f(x) = \frac{1}{x}$ where $a \neq 0$.

- Set up the difference quotient: $\frac{1/x - 1/a}{x - a}$.
- Find a common denominator in the numerator: $\frac{(a-x)/(ax)}{x-a}$.
- Simplify the expression: $\frac{-(x-a)}{ax(x-a)} = -\frac{1}{ax}$.
- Take the limit as $x \rightarrow a$: $\lim_{x \rightarrow a} -\frac{1}{ax} = -\frac{1}{a^2}$.

Answer: $f'(a) = -\frac{1}{a^2}$

2. Let $f(x) = x^{1/3}$. Determine if f is continuous at $x = 0$ and if it is differentiable at $x = 0$.

- Check continuity: $\lim_{x \rightarrow 0} x^{1/3} = 0 = f(0)$, so it is continuous.
- Set up the difference quotient at $a = 0$: $\frac{h^{1/3} - 0}{h}$.
- Simplify the expression: $\frac{h^{1/3}}{h} = \frac{1}{h^{2/3}}$.
- Take the limit as $h \rightarrow 0$: $\lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = \infty$.
- Since the limit is not a finite number, the function is not differentiable at $x = 0$.

Answer: Continuous at $x = 0$, but not differentiable at $x = 0$.

3. Prove that if f is not continuous at a , then it cannot be differentiable at a .

- Use the contrapositive of the theorem 'differentiable implies continuous'.
- The contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$.
- Here, P is 'f is differentiable at a' and Q is 'f is continuous at a'.
- The contrapositive is 'if f is not continuous at a, then f is not differentiable at a'.
- Since the original implication was proven true in the lesson, the contrapositive is also logically true.

Answer: The statement is the contrapositive of the theorem that differentiability implies continuity.