

Module 5 Review and Self-Test

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Practice problems

1. Use the $\epsilon - \delta$ definition to prove that $\lim_{x \rightarrow 2}(x^2 - 1) = 3$.
2. Determine if $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 1]$. Justify your answer.
3. Prove that the function $f(x) = x^3$ is not uniformly continuous on the interval $[0, \infty)$.

Solutions

1. Use the $\epsilon - \delta$ definition to prove that $\lim_{x \rightarrow 2}(x^2 - 1) = 3$.

- We want $|(x^2 - 1) - 3| < \epsilon$, which simplifies to $|x^2 - 4| < \epsilon$.
- Factor the expression: $|x - 2||x + 2| < \epsilon$.
- Assume $\delta \leq 1$, so $|x - 2| < 1$ implies $1 < x < 3$.
- Then $|x + 2| < 5$.
- We need $|x - 2| \cdot 5 < \epsilon$, so $|x - 2| < \epsilon/5$.
- Choose $\delta = \min(1, \epsilon/5)$.

Answer: $\delta = \min(1, \epsilon/5)$

2. Determine if $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 1]$. Justify your answer.

- Check continuity: $f(x) = \sqrt{x}$ is continuous on $[0, 1]$.
- Check domain: The interval $[0, 1]$ is closed and bounded, therefore it is compact.
- Apply Heine-Cantor: Since f is continuous on a compact set, it must be uniformly continuous.

Answer: Yes, it is uniformly continuous by the Heine-Cantor Theorem.

3. Prove that the function $f(x) = x^3$ is not uniformly continuous on the interval $[0, \infty)$.

- We use the negation: $\exists \epsilon > 0$ such that $\forall \delta > 0, \exists x, y$ with $|x - y| < \delta$ but $|x^3 - y^3| \geq \epsilon$.
- Let $\epsilon = 1$. For any $\delta > 0$, we need to find x, y .
- Let $x = n + \delta/2$ and $y = n$. Then $|x - y| = \delta/2 < \delta$.
- The difference is $|(n + \delta/2)^3 - n^3| = |3n^2(\delta/2) + 3n(\delta/2)^2 + (\delta/2)^3|$.
- As $n \rightarrow \infty$, this expression grows without bound for any fixed δ .
- Thus, for large enough n , the difference exceeds 1.

Answer: Not uniformly continuous because the slope increases without bound as $x \rightarrow \infty$.