Monotone functions and their discontinuities; inverse functions

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Practice problems

1. Let $f : [0,1] \to \mathbb{R}$ be a monotone increasing function. Prove that the set $S = \{x \in (0,1) : f(x^+) > f(x^-)\}$ is at most countable.

2. Let $f : [a,b] \to [c,d]$ be a strictly increasing continuous bijection. Prove that f^{-1} is continuous at any point $y_0 \in (c,d)$.

3. Construct a continuous bijection $f : D \to [0,1]$ where D is not an interval, such that f^{-1} is not continuous.

Solutions

1. Let $f : [0,1] \to \mathbb{R}$ be a monotone increasing function. Prove that the set $S = \{x \in (0,1) : f(x^+) > f(x^-)\}$ is at most countable.

   - For each $x \in S$, define the jump interval $J_x = (f(x^-), f(x^+))$.
   - Since f is monotone increasing, if $x < y$, then $f(x^+) \leq f(y^-)$, which implies $J_x \cap J_y = \emptyset$.
   - Each J_x is a non-empty open interval, so it contains at least one rational number q_x.
   - The mapping $x \mapsto q_x$ is an injection from S into $\mathbb{Q}$.
   - Since $\mathbb{Q}$ is countable, S must be at most countable.

   Answer: The set S is countable because it can be injected into the set of rational numbers.

2. Let $f : [a,b] \to [c,d]$ be a strictly increasing continuous bijection. Prove that f^{-1} is continuous at any point $y_0 \in (c,d)$.

   - Let $x_0 = f^{-1}(y_0)$. We want to show that for any $\epsilon > 0$, there exists $\delta > 0$ such that $|y - y_0| < \delta \implies |f^{-1}(y) - x_0| < \epsilon$.
   - Consider the interval $(x_0 - \epsilon, x_0 + \epsilon) \cap [a,b]$. This is an open set in the subspace topology of $[a,b]$.
   - Since f is strictly increasing and continuous on an interval, it maps the open interval $(x_0 - \epsilon, x_0 + \epsilon)$ to an open interval $(f(x_0 - \epsilon), f(x_0 + \epsilon))$.
   - Since $y_0 = f(x_0)$ is inside this open interval, there exists a $\delta > 0$ such that $(y_0 - \delta, y_0 + \delta) \subset (f(x_0 - \epsilon), f(x_0 + \epsilon))$.
   - Thus, if $|y - y_0| < \delta$, then $f^{-1}(y)$ must lie within $(x_0 - \epsilon, x_0 + \epsilon)$, proving continuity.

   Answer: The continuity of f^{-1} follows from the fact that f maps open intervals to open intervals.

3. Construct a continuous bijection $f : D \to [0,1]$ where D is not an interval, such that f^{-1} is not continuous.

   - Let $D = [0, 1/2] \cup [1, 3/2]$.
   - Define $f(x) = x$ for $x \in [0, 1/2]$ and $f(x) = x - 1/2$ for $x \in [1, 3/2]$.
   - The range of the first part is $[0, 1/2]$ and the range of the second part is $[1/2, 1]$.
   - The function is a bijection from D to $[0,1]$ and is continuous on its domain.
   - The inverse is $f^{-1}(y) = y$ for $y \in [0, 1/2)$ and $f^{-1}(y) = y + 1/2$ for $y \in [1/2, 1]$.
   - As $y \to 1/2$ from the left, $f^{-1}(y) \to 1/2$, but $f^{-1}(1/2) = 1$. The jump at $y = 1/2$ proves discontinuity.

   Answer: A function defined on $D = [0, 1/2] \cup [1, 3/2]$ with $f(x) = x$ on the first part and $f(x) = x - 1/2$ on the second part works.