

Uniform continuity II — Heine-Cantor

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Practice problems

1. Use the Heine-Cantor Theorem to determine if $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 1]$.
2. Explain why the Heine-Cantor Theorem cannot be used to prove that $f(x) = \frac{1}{x}$ is uniformly continuous on the interval $(0, 1)$.
3. Let f be a function defined on $[a, b]$. If f is continuous on (a, b) but not uniformly continuous on (a, b) , what can we conclude about the continuity of f at the endpoints a and b ?

Solutions

1. Use the Heine-Cantor Theorem to determine if $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 1]$.
 - Check if $f(x) = \sqrt{x}$ is continuous on $[0, 1]$. Since the square root function is continuous for all $x \geq 0$, it is continuous on $[0, 1]$.
 - Check if the domain $[0, 1]$ is compact. The interval is closed and bounded, so it is compact.
 - Apply the Heine-Cantor Theorem: a continuous function on a compact set is uniformly continuous.

Answer: Yes, it is uniformly continuous.

2. Explain why the Heine-Cantor Theorem cannot be used to prove that $f(x) = \frac{1}{x}$ is uniformly continuous on the interval $(0, 1)$.
 - The function $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$.
 - The interval $(0, 1)$ is bounded, but it is not closed because the endpoints 0 and 1 are not included.
 - Since the domain is not closed, it is not compact.
 - The Heine-Cantor Theorem requires a compact domain to guarantee uniform continuity.

Answer: The domain $(0, 1)$ is not compact because it is not closed.

3. Let f be a function defined on $[a, b]$. If f is continuous on (a, b) but not uniformly continuous on (a, b) , what can we conclude about the continuity of f at the endpoints a and b ?
 - If f were continuous on the closed interval $[a, b]$, then by the Heine-Cantor Theorem, it would be uniformly continuous on $[a, b]$.
 - If a function is uniformly continuous on a set, it is also uniformly continuous on any subset of that set.
 - Therefore, if f were continuous on $[a, b]$, it would necessarily be uniformly continuous on (a, b) .
 - Since we are given that f is NOT uniformly continuous on (a, b) , the hypothesis that f is continuous on $[a, b]$ must be false.
 - This means f must be discontinuous at at least one of the endpoints a or b .

Answer: The function f must be discontinuous at at least one of the endpoints a or b .