

Uniform continuity I — the definition and why it's different

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Practice problems

1. Determine if $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 1]$.
2. Prove that $f(x) = 3x - 2$ is uniformly continuous on $\mathbb{R}$.
3. Show that $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

Solutions

1. Determine if $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 1]$.

- We want to see if for any $\epsilon > 0$, we can find a δ such that $|x - y| < \delta$ implies $|\sqrt{x} - \sqrt{y}| < \epsilon$.
- A useful property for square roots on $[0, 1]$ is that $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$.
- If we choose $\delta = \epsilon^2$, then $|x - y| < \delta$ implies $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|} < \sqrt{\epsilon^2} = \epsilon$.
- Since this δ depends only on ϵ and not on x or y , the function is uniformly continuous.

Answer: Yes, it is uniformly continuous.

2. Prove that $f(x) = 3x - 2$ is uniformly continuous on $\mathbb{R}$.

- Consider $|f(x) - f(y)| = |(3x - 2) - (3y - 2)| = |3x - 3y| = 3|x - y|$.
- We want $3|x - y| < \epsilon$.
- This is satisfied if $|x - y| < \epsilon/3$.
- Since $\delta = \epsilon/3$ depends only on ϵ and not on x or y , the function is uniformly continuous.

Answer: $\delta = \epsilon/3$

3. Show that $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

- Pick $\epsilon = 1$. We must show that for any $\delta > 0$, there exist x, y such that $|x - y| < \delta$ but $|x^3 - y^3| \geq 1$.
- Let $x = n + \delta/2$ and $y = n$. Then $|x - y| = \delta/2 < \delta$.
- The difference is $|x^3 - y^3| = |(n + \delta/2)^3 - n^3| = |3n^2(\delta/2) + 3n(\delta/2)^2 + (\delta/2)^3|$.
- As $n \rightarrow \infty$, the term $3n^2(\delta/2)$ grows without bound and will eventually exceed 1 for any fixed δ .
- Therefore, no global δ exists for $\epsilon = 1$.

Answer: Not uniformly continuous.