

The Intermediate Value Theorem — proved

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Practice problems

1. Show that the equation $x^5 + x + 1 = 0$ has at least one real root in the interval $[-2, 0]$.
2. Let f be a continuous function on $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$. Prove there exists $c \in [0, 1]$ such that $f(c) = c$.
3. Suppose f is continuous on $[0, 1]$ and $f(0) = f(1)$. Prove that for any y between $f(0)$ and the maximum value of f , there are at least two distinct points c_1, c_2 such that $f(c_1) = f(c_2) = y$.

Solutions

1. Show that the equation $x^5 + x + 1 = 0$ has at least one real root in the interval $[-2, 0]$.

- Define $f(x) = x^5 + x + 1$.
- Since f is a polynomial, it is continuous on $[-2, 0]$.
- Calculate $f(-2) = (-2)^5 + (-2) + 1 = -32 - 2 + 1 = -33$.
- Calculate $f(0) = 0^5 + 0 + 1 = 1$.
- Since $f(-2) < 0 < f(0)$, the Intermediate Value Theorem guarantees a $c \in (-2, 0)$ such that $f(c) = 0$.

Answer: A root exists because $f(-2) = -33$ and $f(0) = 1$.

2. Let f be a continuous function on $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$. Prove there exists $c \in [0, 1]$ such that $f(c) = c$.

- Define a helper function $g(x) = f(x) - x$.
- Since f and x are continuous, g is continuous on $[0, 1]$.
- Calculate $g(0) = f(0) - 0 = 0 - 0 = 0$.
- Calculate $g(1) = f(1) - 1 = 1 - 1 = 0$.
- In this specific case, $g(0) = 0$ and $g(1) = 0$, so $c = 0$ and $c = 1$ are already solutions. If we wanted a point in $(0, 1)$, we would need different endpoint values, but the theorem holds for the closed interval.

Answer: The points $c = 0$ and $c = 1$ satisfy $f(c) = c$.

3. Suppose f is continuous on $[0, 1]$ and $f(0) = f(1)$. Prove that for any y between $f(0)$ and the maximum value of f , there are at least two distinct points c_1, c_2 such that $f(c_1) = f(c_2) = y$.

- Let M be the maximum value of f on $[0, 1]$, attained at $x = d$ by the Extreme Value Theorem.
- If d is an endpoint, the result is trivial or y cannot be between $f(0)$ and M . Assume $d \in (0, 1)$.
- Consider the interval $[0, d]$. Since $f(0) < y < f(d)$, the IVT guarantees $c_1 \in (0, d)$ such that $f(c_1) = y$.
- Consider the interval $[d, 1]$. Since $f(d) > y > f(1) = f(0)$, the IVT guarantees $c_2 \in (d, 1)$ such that $f(c_2) = y$.
- Since $c_1 < d < c_2$, the points are distinct.

Answer: The IVT applied to the two sub-intervals $[0, d]$ and $[d, 1]$ guarantees two distinct points.