

Continuity and compactness: the Extreme Value Theorem — proved

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Practice problems

1. Consider the function $f(x) = 1/x$ on the interval $(0, 1]$. Does this function attain an absolute maximum or minimum on this domain? Explain why the Extreme Value Theorem does not guarantee these values.
2. Let $f(x) = x^2 - 4x + 5$ on the interval $[0, 3]$. Use the Extreme Value Theorem to justify that extrema exist, and then find the absolute maximum and minimum values.
3. Suppose f is continuous on $[a, b]$ and $f(a) = f(b)$. Prove that if f is not constant, it must attain a value strictly greater than $f(a)$ or strictly less than $f(a)$.

Solutions

1. Consider the function $f(x) = 1/x$ on the interval $(0, 1]$. Does this function attain an absolute maximum or minimum on this domain? Explain why the Extreme Value Theorem does not guarantee these values.

- The interval $(0, 1]$ is not closed because it does not contain its limit point 0.
- The Extreme Value Theorem requires a closed and bounded interval $[a, b]$.
- As $x \rightarrow 0^+$, $f(x) \rightarrow \infty$, so the function is not bounded above and has no absolute maximum.
- At $x = 1$, $f(1) = 1$. Since $1/x \geq 1$ for all $x \in (0, 1]$, the absolute minimum is 1.

Answer: No absolute maximum exists; absolute minimum is 1 at $x = 1$. EVT fails because the interval is not closed.

2. Let $f(x) = x^2 - 4x + 5$ on the interval $[0, 3]$. Use the Extreme Value Theorem to justify that extrema exist, and then find the absolute maximum and minimum values.

- The function is a polynomial (continuous) and the interval $[0, 3]$ is closed and bounded, so EVT applies.
- Find critical points: $f'(x) = 2x - 4 = 0 \implies x = 2$.
- Evaluate at endpoints and critical point: $f(0) = 5$, $f(2) = 4 - 8 + 5 = 1$, $f(3) = 9 - 12 + 5 = 2$.
- Comparing values: the maximum is 5 and the minimum is 1.

Answer: Absolute maximum is 5 at $x = 0$; absolute minimum is 1 at $x = 2$.

3. Suppose f is continuous on $[a, b]$ and $f(a) = f(b)$. Prove that if f is not constant, it must attain a value strictly greater than $f(a)$ or strictly less than $f(a)$.

- By the Extreme Value Theorem, f attains an absolute maximum M and an absolute minimum m on $[a, b]$.
- Since f is not constant, it cannot be that $M = m$.
- If $M = m$, then $f(x) = M$ for all x , which contradicts that f is not constant.
- Since $f(a) = f(b)$, if $M = f(a)$, then m must be strictly less than $f(a)$.
- If $m = f(a)$, then M must be strictly greater than $f(a)$.
- If neither M nor m equals $f(a)$, then both are different from $f(a)$.

Answer: Since $M > m$ and $f(a)$ is just one value, at least one of the extrema must differ from $f(a)$.