

Algebra of continuous functions; compositions

Lesson 60 · Module 5 — Limits & Continuity · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Determine the domain of continuity for the function $f(x) = \frac{\sqrt{x+5}}{x^2-9}$.
2. Let $g(x) = e^x$ and $f(x) = \frac{1}{x-1}$. Is the composition $(f \circ g)(x)$ continuous at $x = 0$? Explain why or why not.
3. Prove that any function of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous for all $x \in \mathbb{R}$ using the algebra of continuity.

Solutions

1. Determine the domain of continuity for the function $f(x) = \frac{\sqrt{x+5}}{x^2-9}$.

- The numerator $\sqrt{x+5}$ is continuous where $x+5 \geq 0$, which is $x \geq -5$.
- The denominator $x^2 - 9$ is a polynomial and thus continuous everywhere.
- The quotient is continuous where both are continuous and the denominator is non-zero.
- Solve $x^2 - 9 = 0$ to find $x = 3$ and $x = -3$.
- Combine the conditions: $x \geq -5$ and $x \neq 3, -3$.

Answer: $[-5, -3) \cup (-3, 3) \cup (3, \infty)$

2. Let $g(x) = e^x$ and $f(x) = \frac{1}{x-1}$. Is the composition $(f \circ g)(x)$ continuous at $x = 0$? Explain why or why not.

- Check the inner function $g(x) = e^x$ at $x = 0$. g is continuous everywhere, so it is continuous at 0.
- Calculate the value $g(0) = e^0 = 1$.
- Check the outer function $f(u) = \frac{1}{u-1}$ at $u = g(0) = 1$.
- The function f has a vertical asymptote at $u = 1$ and is therefore discontinuous there.
- Since f is not continuous at $g(0)$, the composition is not continuous at $x = 0$.

Answer: No, because the outer function is discontinuous at $g(0) = 1$.

3. Prove that any function of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous for all $x \in \mathbb{R}$ using the algebra of continuity.

- The constant function $f(x) = a_i$ is continuous everywhere.
- The identity function $f(x) = x$ is continuous everywhere.
- By the product rule, x^k (the product of x with itself k times) is continuous everywhere.
- By the product rule again, $a_k x^k$ is continuous everywhere.
- A polynomial is a finite sum of such terms $a_k x^k$.
- By the sum rule, the sum of continuous functions is continuous.

Answer: Polynomials are continuous as they are finite sums of products of continuous functions.