

The pathological gallery: Dirichlet, Thomae

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Practice problems

1. Let $g(x) = x$ if x is rational and $g(x) = 0$ if x is irrational. At which points is $g(x)$ continuous?
2. Prove that the Dirichlet function $f(x)$ is discontinuous at $x = \pi$.
3. For Thomae's function, show that $\lim_{x \rightarrow a} f(x) = 0$ for any $a \in \mathbb{R}$.

Solutions

1. Let $g(x) = x$ if x is rational and $g(x) = 0$ if x is irrational. At which points is $g(x)$ continuous?

- If $x = 0$, then $g(0) = 0$. For any sequence $x_n \rightarrow 0$, if x_n is rational, $g(x_n) = x_n \rightarrow 0$. If x_n is irrational, $g(x_n) = 0 \rightarrow 0$. Thus g is continuous at $x = 0$.
- If $x \neq 0$ and x is rational, $g(x) = x$. A sequence of irrationals $x_n \rightarrow x$ gives $g(x_n) = 0 \rightarrow 0 \neq x$. Discontinuous.
- If $x \neq 0$ and x is irrational, $g(x) = 0$. A sequence of rationals $x_n \rightarrow x$ gives $g(x_n) = x_n \rightarrow x \neq 0$. Discontinuous.

Answer: Continuous only at $x = 0$.

2. Prove that the Dirichlet function $f(x)$ is discontinuous at $x = \pi$.

- The value of the function at $x = \pi$ is $f(\pi) = 0$ because π is irrational.
- Since the rationals are dense, there exists a sequence of rational numbers q_n such that $q_n \rightarrow \pi$.
- For each q_n , $f(q_n) = 1$.
- The limit of the images is $\lim_{n \rightarrow \infty} f(q_n) = 1$.
- Since $1 \neq 0$, the function is discontinuous at $x = \pi$.

Answer: The limit of a rational sequence is 1, while $f(\pi) = 0$.

3. For Thomae's function, show that $\lim_{x \rightarrow a} f(x) = 0$ for any $a \in \mathbb{R}$.

- Fix $\epsilon > 0$. We need to find $\delta > 0$ such that $|x - a| < \delta \implies |f(x)| < \epsilon$.
- The condition $f(x) \geq \epsilon$ only happens if $x = p/q$ with $q \leq 1/\epsilon$.
- In any bounded interval around a , say $(a - 1, a + 1)$, there are only finitely many such rationals.
- Let these rationals be $r_1, r_2, \dots, r_k$.
- Choose $\delta = \min(|a - r_1|, \dots, |a - r_k|)$ if a is not one of them, or a smaller value if it is.
- Then for all x in the δ -window, $f(x) < \epsilon$.

Answer: The limit is 0 because rationals with small denominators are finite in any bounded region.