

Continuity: Three Equivalent Definitions

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Practice problems

1. Use the sequential definition to determine if the function $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = x^2$ for $x \notin \mathbb{Q}$ is continuous at $x = 0$.
2. Prove using the epsilon-delta definition that $f(x) = 3x + 5$ is continuous at $x = 1$.
3. Let $f(x) = 0$ for $x < 1$ and $f(x) = 1$ for $x \geq 1$. Use the open-set definition to show that f is discontinuous at $x = 1$.

Solutions

1. Use the sequential definition to determine if the function $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = x^2$ for $x \notin \mathbb{Q}$ is continuous at $x = 0$.

- Check the function value at the point: $f(0) = 0$ since 0 is rational.
- Consider any sequence $x_n \rightarrow 0$.
- If x_n is rational, $f(x_n) = x_n$, which converges to 0.
- If x_n is irrational, $f(x_n) = x_n^2$, which also converges to 0.
- Since $f(x_n) \rightarrow 0$ for all sequences $x_n \rightarrow 0$, the function is continuous at $x = 0$.

Answer: Continuous at $x = 0$

2. Prove using the epsilon-delta definition that $f(x) = 3x + 5$ is continuous at $x = 1$.

- We want to show that for any $\epsilon > 0$, there exists $\delta > 0$ such that $|x - 1| < \delta \implies |(3x + 5) - (3(1) + 5)| < \epsilon$.
- Simplify the output expression: $|3x + 5 - 8| = |3x - 3| = 3|x - 1|$.
- We need $3|x - 1| < \epsilon$, which means $|x - 1| < \epsilon/3$.
- Choose $\delta = \epsilon/3$.
- Then $|x - 1| < \delta \implies 3|x - 1| < 3(\epsilon/3) = \epsilon$.

Answer: $\delta = \epsilon/3$

3. Let $f(x) = 0$ for $x < 1$ and $f(x) = 1$ for $x \geq 1$. Use the open-set definition to show that f is discontinuous at $x = 1$.

- The function value is $f(1) = 1$.
- Pick an open set V containing 1, for example $V = (0.5, 1.5)$.
- Find the preimage $f^{-1}(V)$. This is the set of all x such that $f(x) \in (0.5, 1.5)$.
- Since the only possible values of $f(x)$ are 0 and 1, we need $f(x) = 1$.
- This occurs for $x \in [1, \infty)$.
- The set $[1, \infty)$ is not open because it contains its left endpoint 1 but no open neighborhood around 1 is contained in the set.
- Since the preimage of an open set is not open, f is discontinuous at $x = 1$.

Answer: Discontinuous at $x = 1$