

eps-delta proof clinic II — the standard tricks

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Practice problems

1. Prove using the epsilon-delta definition that $\lim_{x \rightarrow 3}(x^2 - x) = 6$.
2. Prove that $\lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2}$.
3. Prove that $\lim_{x \rightarrow 2}(x^2 + 2x) = 8$.

Solutions

1. Prove using the epsilon-delta definition that $\lim_{x \rightarrow 3}(x^2 - x) = 6$.

- Consider $|(x^2 - x) - 6| = |x^2 - x - 6| = |x - 3||x + 2|$.
- Assume $\delta \leq 1$. Then $2 < x < 4$, so $|x + 2| < 6$.
- We want $|x - 3||x + 2| < 6|x - 3| < \epsilon$, which requires $|x - 3| < \epsilon/6$.
- Choose $\delta = \min(1, \epsilon/6)$.
- Then $|x^2 - x - 6| \leq 6|x - 3| < 6(\epsilon/6) = \epsilon$.

Answer: $\delta = \min(1, \frac{\epsilon}{6})$

2. Prove that $\lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2}$.

- Consider $|\frac{1}{x+1} - \frac{1}{2}| = |\frac{2-(x+1)}{2(x+1)}| = \frac{|1-x|}{|2(x+1)|} = \frac{|x-1|}{|2(x+1)|}$.
- Assume $\delta \leq 1$. Then $0 < x < 2$, so $x + 1 > 1$, meaning $|2(x+1)| > 2$.
- Thus $\frac{1}{|2(x+1)|} < \frac{1}{2}$.
- We want $\frac{1}{2}|x - 1| < \epsilon$, so $|x - 1| < 2\epsilon$.
- Choose $\delta = \min(1, 2\epsilon)$.

Answer: $\delta = \min(1, 2\epsilon)$

3. Prove that $\lim_{x \rightarrow 2}(x^2 + 2x) = 8$.

- Consider $|x^2 + 2x - 8| = |x - 2||x + 4|$.
- Assume $\delta \leq 1$. Then $1 < x < 3$, so $|x + 4| < 7$.
- We want $7|x - 2| < \epsilon$, so $|x - 2| < \epsilon/7$.
- Choose $\delta = \min(1, \epsilon/7)$.
- Then $|x^2 + 2x - 8| \leq 7|x - 2| < 7(\epsilon/7) = \epsilon$.

Answer: $\delta = \min(1, \frac{\epsilon}{7})$