

Functional limits: eps-delta in full

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Practice problems

1. Use the $\epsilon - \delta$ definition to prove that $\lim_{x \rightarrow 4} (2x + 3) = 11$.
2. Use the $\epsilon - \delta$ definition to prove that $\lim_{x \rightarrow 2} (5x - 7) = 3$.
3. Prove that $\lim_{x \rightarrow 1} x^2 = 1$ using the $\epsilon - \delta$ definition.

Solutions

1. Use the $\epsilon - \delta$ definition to prove that $\lim_{x \rightarrow 4}(2x + 3) = 11$.

- Step 1: Scratch work. We want $|(2x + 3) - 11| < \epsilon$.
- Step 2: Simplify the expression: $|2x - 8| = 2|x - 4|$.
- Step 3: Set $2|x - 4| < \epsilon$, which implies $|x - 4| < \epsilon/2$.
- Step 4: Formal proof. Let $\epsilon > 0$. Choose $\delta = \epsilon/2$.
- Step 5: If $0 < |x - 4| < \delta$, then $|(2x + 3) - 11| = 2|x - 4| < 2\delta = 2(\epsilon/2) = \epsilon$.

Answer: $\delta = \epsilon/2$

2. Use the $\epsilon - \delta$ definition to prove that $\lim_{x \rightarrow 2}(5x - 7) = 3$.

- Step 1: Scratch work. $|(5x - 7) - 3| = |5x - 10| = 5|x - 2|$.
- Step 2: We want $5|x - 2| < \epsilon$, so $|x - 2| < \epsilon/5$.
- Step 3: Formal proof. Let $\epsilon > 0$. Choose $\delta = \epsilon/5$.
- Step 4: If $0 < |x - 2| < \delta$, then $|(5x - 7) - 3| = 5|x - 2| < 5\delta = \epsilon$.

Answer: $\delta = \epsilon/5$

3. Prove that $\lim_{x \rightarrow 1} x^2 = 1$ using the $\epsilon - \delta$ definition.

- Step 1: Scratch work. $|x^2 - 1| = |x - 1||x + 1|$.
- Step 2: Bound $|x + 1|$. Assume $\delta \leq 1$. Then $|x - 1| < 1 \implies 0 < x < 2 \implies |x + 1| < 3$.
- Step 3: We want $|x - 1||x + 1| < 3|x - 1| < \epsilon$, so $|x - 1| < \epsilon/3$.
- Step 4: Choose $\delta = \min(1, \epsilon/3)$.
- Step 5: If $0 < |x - 1| < \delta$, then $|x^2 - 1| = |x - 1||x + 1| < \delta \cdot 3 \leq (\epsilon/3) \cdot 3 = \epsilon$.

Answer: $\delta = \min(1, \epsilon/3)$