

Module 4 Review and Self-Test

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Practice problems

1. Determine if the set $S = \bigcup_{n=1}^{\infty} (\frac{1}{2n}, \frac{1}{2n-1})$ is open, closed, or neither.
2. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Find the derived set S' and the closure $\overline{S}$.
3. Is the set $S = [0, 1] \cup [2, 3]$ compact? Is it connected?

Solutions

1. Determine if the set $S = \bigcup_{n=1}^{\infty} (\frac{1}{2n}, \frac{1}{2n-1})$ is open, closed, or neither.
- Every point in S belongs to at least one of the open intervals $(\frac{1}{2n}, \frac{1}{2n-1})$.
 - Since each such interval is open, every point in S is an interior point.
 - Therefore, S is open.

Answer: Open

2. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Find the derived set S' and the closure $\overline{S}$.
- The sequence $1/n$ converges to 0.
 - Any open interval around 0 contains infinitely many points of S .
 - No other point in $\mathbb{R}$ is a limit point because all other points are isolated.
 - Thus, $S' = \{0\}$.
 - The closure is $S \cup S' = \{1, 1/2, 1/3, \dots, 0\}$.

Answer: $S' = \{0\}$, $\overline{S} = \{1/n : n \in \mathbb{N}\} \cup \{0\}$

3. Is the set $S = [0, 1] \cup [2, 3]$ compact? Is it connected?
- The set is a union of two closed intervals, so it is closed.
 - The set is contained within the interval $[0, 3]$, so it is bounded.
 - Since it is closed and bounded, it is compact by Heine-Borel.
 - The set is not an interval because there is a gap between 1 and 2.
 - Therefore, it is not connected.

Answer: Compact, but not connected