

Why topology matters — a preview

Lesson 53 · Module 4 — Topology of the Real Line · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Let $f : (0, 1) \rightarrow \mathbb{R}$ be defined by $f(x) = \sin(1/x)$. Is the domain $(0, 1)$ compact? Is the function f bounded on this domain?
2. Suppose f is a continuous function on the set $S = [0, 1] \cup [2, 3]$. If $f(1) = 5$ and $f(2) = 10$, must there be a point $c \in S$ such that $f(c) = 7$?
3. Explain why the function $f(x) = \frac{1}{x}$ is not uniformly continuous on the interval $(0, 1)$.

Solutions

1. Let $f : (0, 1) \rightarrow \mathbb{R}$ be defined by $f(x) = \sin(1/x)$. Is the domain $(0, 1)$ compact? Is the function f bounded on this domain?

- The domain $(0, 1)$ is bounded but not closed, so by the Heine-Borel Theorem, it is not compact.
- The function $f(x) = \sin(1/x)$ takes values in the range $[-1, 1]$ for all x in the domain.
- Therefore, the function is bounded, even though the domain is not compact.

Answer: The domain is not compact, but the function is bounded.

2. Suppose f is a continuous function on the set $S = [0, 1] \cup [2, 3]$. If $f(1) = 5$ and $f(2) = 10$, must there be a point $c \in S$ such that $f(c) = 7$?

- The Intermediate Value Theorem requires the domain to be connected.
- The set $S = [0, 1] \cup [2, 3]$ is not connected because it is the union of two disjoint closed sets.
- Since the domain is disconnected, the function can jump from 5 to 10 without passing through 7.
- For example, $f(x) = 5$ on $[0, 1]$ and $f(x) = 10$ on $[2, 3]$ is continuous on S but never equals 7.

Answer: No, because the domain is not connected.

3. Explain why the function $f(x) = \frac{1}{x}$ is not uniformly continuous on the interval $(0, 1)$.

- For uniform continuity, a single δ must work for all x, y in the domain.
- As x approaches 0, the slope of $1/x$ becomes arbitrarily steep.
- This means for a fixed ϵ , the required δ must become smaller and smaller as we move toward 0.
- Since there is no single positive δ that works for the entire interval, the function is not uniformly continuous.
- Note that the domain $(0, 1)$ is not compact, which is why the theorem continuous on compact implies uniformly continuous does not apply.

Answer: The slope increases without bound near 0, requiring δ to approach 0, so no global δ exists.