

Dense sets and G-delta/F-sigma — a glimpse

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Practice problems

1. Let $S = \{\frac{m}{3^n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$. Prove that S is dense in $\mathbb{R}$.
2. Is the set of all integers $\mathbb{Z}$ an F-sigma set? Explain why or why not.
3. If S is a G-delta set, what can you say about its complement S^c ? Prove it using the definitions.

Solutions

1. Let $S = \{\frac{m}{3^n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$. Prove that S is dense in $\mathbb{R}$.

- Consider any interval (a, b) with $a < b$.
- The width of the interval is $\epsilon = b - a > 0$.
- Choose n such that $1/3^n < \epsilon$.
- The distance between consecutive points $m/3^n$ and $(m+1)/3^n$ is $1/3^n$.
- Since the gap is smaller than the interval width, there must be at least one m such that $a < m/3^n < b$.

Answer: The set S is dense because for any interval (a, b) , we can find n such that $1/3^n < b - a$, ensuring a point of S lies within the interval.

2. Is the set of all integers $\mathbb{Z}$ an F-sigma set? Explain why or why not.

- The set of integers $\mathbb{Z}$ is countable.
- We can write $\mathbb{Z} = \bigcup_{k \in \mathbb{Z}} \{k\}$.
- Each singleton set $\{k\}$ is a closed set in $\mathbb{R}$.
- Since $\mathbb{Z}$ is a countable union of closed sets, it satisfies the definition of an F-sigma set.

Answer: Yes, $\mathbb{Z}$ is F-sigma because it is a countable union of closed singleton sets.

3. If S is a G-delta set, what can you say about its complement S^c ? Prove it using the definitions.

- By definition, $S = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open.
- The complement is $S^c = (\bigcap_{n=1}^{\infty} G_n)^c$.
- By De Morgan's Law, the complement of an intersection is the union of the complements: $S^c = \bigcup_{n=1}^{\infty} G_n^c$.
- Since each G_n is open, each G_n^c is closed.
- Thus, S^c is a countable union of closed sets, which is the definition of an F-sigma set.

Answer: The complement of a G-delta set is always an F-sigma set.