

Perfect sets and the Cantor set revisited

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Practice problems

1. Let $S = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Is S a perfect set? Justify your answer.
2. Prove that the set $S = [0, 1] \cup [2, 3]$ is a perfect set.
3. Consider the Cantor ternary set C . We know it is perfect. Use this to explain why the Cantor set must be uncountable, and explain why this does not contradict the fact that it contains no intervals.

Solutions

1. Let $S = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Is S a perfect set? Justify your answer.

- A set is perfect if it is closed and has no isolated points.
- The set S is closed because its only limit point is 0, and 0 is in S .
- However, consider the point $x = 1$.
- For $\epsilon = 0.1$, the interval $(0.9, 1.1)$ contains no points of S other than 1 itself.
- Thus, 1 is an isolated point.
- Since S contains an isolated point, it is not perfect.

Answer: No, S is not perfect because it contains isolated points.

2. Prove that the set $S = [0, 1] \cup [2, 3]$ is a perfect set.

- A set is perfect if it is closed and every point is a limit point.
- The union of two closed intervals is closed.
- For any $x \in [0, 1]$, any interval $(x - \epsilon, x + \epsilon)$ contains other points of $[0, 1]$, so x is a limit point.
- Similarly, for any $x \in [2, 3]$, any interval $(x - \epsilon, x + \epsilon)$ contains other points of $[2, 3]$, so x is a limit point.
- Since all points are limit points and the set is closed, S is perfect.

Answer: The set is closed and contains no isolated points, thus it is perfect.

3. Consider the Cantor ternary set C . We know it is perfect. Use this to explain why the Cantor set must be uncountable, and explain why this does not contradict the fact that it contains no intervals.

- Since C is a non-empty perfect set, the theorem proved in the lesson states it must be uncountable.
- Uncountability refers to the cardinality (the number of elements), not the measure (the length).
- A set can be uncountable without containing any intervals; this is the definition of being totally disconnected.
- The Cantor set is the classic example of a set with cardinality $2^{\aleph_0}$ but length zero.

Answer: Perfect sets are always uncountable; uncountability does not imply the existence of an interval.