

Connectedness — intervals are the only connected sets

Lesson 50 · Module 4 — Topology of the Real Line · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Determine if the set $S = (0, 2) \cup [1, 3]$ is connected. Justify your answer.
2. Is the set $S = \mathbb{R} \setminus \{0, 1\}$ connected? If not, provide a separation.
3. Prove that any finite set with more than one element is disconnected.

Solutions

1. Determine if the set $S = (0, 2) \cup [1, 3]$ is connected. Justify your answer.

- Step 1: Simplify the union. Since the intervals $(0, 2)$ and $[1, 3]$ overlap (for example, the point 1.5 is in both), their union is a single interval.
- Step 2: The union is $(0, 3]$.
- Step 3: By the theorem proved in class, every interval is connected.
- Step 4: Therefore, S is connected.

Answer: Connected

2. Is the set $S = \mathbb{R} \setminus \{0, 1\}$ connected? If not, provide a separation.

- Step 1: The set S consists of three disjoint pieces: $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.
- Step 2: To show it is disconnected, we only need one separation. Let $U = (-\infty, 0.5)$ and $V = (0.5, \infty)$.
- Step 3: U and V are open sets. Their union is $\mathbb{R} \setminus \{0.5\}$, which contains S .
- Step 4: Their intersection is empty.
- Step 5: $S \cap U$ contains -1 and $S \cap V$ contains 2 , so both are non-empty.
- Step 6: Thus, S is disconnected.

Answer: Disconnected

3. Prove that any finite set with more than one element is disconnected.

- Step 1: Let $S = \{x_1, x_2, \dots, x_n\}$ with $n > 1$. Assume the elements are ordered $x_1 < x_2 < \dots < x_n$.
- Step 2: We can pick a value r such that $x_1 < r < x_2$.
- Step 3: Define $U = (-\infty, r)$ and $V = (r, \infty)$.
- Step 4: U and V are open and their union is $\mathbb{R} \setminus \{r\}$, which contains S .
- Step 5: $U \cap V = \emptyset$.
- Step 6: $x_1 \in S \cap U$ and $x_2 \in S \cap V$, so both are non-empty.
- Step 7: Therefore, S is disconnected.

Answer: Disconnected