

Compactness III — sequential compactness; the three faces unified

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Practice problems

1. Determine if the set $S = [0, 1) \cup \{2\}$ is sequentially compact. Justify your answer using the Heine-Borel theorem.
2. Let $S = \{x \in \mathbb{R} : 1 \leq x^2 \leq 9\}$. Is S sequentially compact?
3. Consider the set $S = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$. Is S sequentially compact? If not, what is the smallest set S' containing S that is sequentially compact?

Solutions

1. Determine if the set $S = [0, 1) \cup \{2\}$ is sequentially compact. Justify your answer using the Heine-Borel theorem.

- Check if the set is bounded: S is contained within $[0, 2]$, so it is bounded.
- Check if the set is closed: The point 1 is a limit point of S because any neighborhood of 1 contains points from $[0, 1)$.
- However, 1 is not in S .
- Since S does not contain all its limit points, it is not closed.
- By Heine-Borel, since it is not closed, it is not compact and therefore not sequentially compact.

Answer: Not sequentially compact.

2. Let $S = \{x \in \mathbb{R} : 1 \leq x^2 \leq 9\}$. Is S sequentially compact?

- Solve the inequality $1 \leq x^2 \leq 9$.
- This implies $x \in [-3, -1] \cup [1, 3]$.
- The set S is a union of two closed intervals.
- It is bounded (contained in $[-3, 3]$) and closed (union of closed sets).
- By the Heine-Borel theorem, S is compact, and thus sequentially compact.

Answer: Yes, it is sequentially compact.

3. Consider the set $S = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$. Is S sequentially compact? If not, what is the smallest set S' containing S that is sequentially compact?

- Check boundedness: S is contained in $[1, 2]$, so it is bounded.
- Check closedness: The sequence $1 + \frac{1}{n}$ converges to 1.
- The point 1 is a limit point of S , but 1 is not in S (since $1 + \frac{1}{n} > 1$ for all n).
- Thus S is not closed and not sequentially compact.
- To make it sequentially compact, we must add all limit points. The only limit point is 1.
- The set $S' = S \cup \{1\}$ is closed and bounded, hence sequentially compact.

Answer: S is not sequentially compact; $S' = \{1 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\}$.