

# Compactness II — Heine-Borel

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## Practice problems

1. Determine if the set  $S = \{x \in \mathbb{R} : \frac{1}{x} \geq 1\}$  is compact.
2. Is the set  $S = \{x \in \mathbb{R} : x^2 \geq 1\}$  compact?
3. Let  $S$  be the set of all  $x$  such that  $x = 1 + \frac{1}{n}$  for some  $n \in \mathbb{N}$ . Is  $S$  compact?

## Solutions

1. Determine if the set  $S = \{x \in \mathbb{R} : \frac{1}{x} \geq 1\}$  is compact.

- First, solve the inequality  $\frac{1}{x} \geq 1$ . For this to be true,  $x$  must be positive, so  $1 \geq x$ , which means  $0 < x \leq 1$ .
- The set is  $S = (0, 1]$ .
- Check boundedness: The set is contained in  $[0, 1]$ , so it is bounded.
- Check closedness: The point 0 is a limit point of  $S$ , but 0 is not in  $S$ . Therefore,  $S$  is not closed.
- Since  $S$  is not closed, it is not compact by the Heine-Borel Theorem.

**Answer:** Not compact

2. Is the set  $S = \{x \in \mathbb{R} : x^2 \geq 1\}$  compact?

- Solve the inequality  $x^2 \geq 1$ . This means  $x \geq 1$  or  $x \leq -1$ .
- The set is  $S = (-\infty, -1] \cup [1, \infty)$ .
- Check boundedness: The set extends to infinity in both directions, so it is not bounded.
- Since it is not bounded, it cannot be compact by the Heine-Borel Theorem.

**Answer:** Not compact

3. Let  $S$  be the set of all  $x$  such that  $x = 1 + \frac{1}{n}$  for some  $n \in \mathbb{N}$ . Is  $S$  compact?

- The elements of  $S$  are  $2, 1.5, 1.33, \dots$  and they converge to 1.
- Check boundedness: All elements are between 1 and 2, so  $S$  is bounded.
- Check closedness: The only limit point of the sequence  $1 + \frac{1}{n}$  is 1.
- However, 1 is not in the set  $S$  because  $1 + \frac{1}{n}$  is always strictly greater than 1 for any natural number  $n$ .
- Since the limit point 1 is missing,  $S$  is not closed.
- Therefore,  $S$  is not compact.

**Answer:** Not compact