# Compactness I — open covers; why [0,1] is special



## Practice problems

1. Determine if the set $S = [1, 2]$ is compact. Justify your answer using the closed and bounded criteria.
2. Consider the set $S = (0, 1]$. Is this set compact? If not, provide an open cover that has no finite subcover.
3. Is the set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ compact? Hint: Consider the limit points of the set.

## Solutions

1. Determine if the set $S = [1, 2]$ is compact. Justify your answer using the closed and bounded criteria.

   - Step 1: Check if $S$ is bounded. Since $1 \leq x \leq 2$ for all $x \in S$, the set is contained within the interval $[-10, 10]$, so it is bounded.
   - Step 2: Check if $S$ is closed. The set $[1, 2]$ contains all its limit points (the limit points are the interval itself), so it is closed.
   - Step 3: Since $S$ is both closed and bounded in $\mathbb{R}$, it is compact by the Heine-Borel theorem.

   **Answer:** Yes, it is compact.

2. Consider the set $S = (0, 1]$. Is this set compact? If not, provide an open cover that has no finite subcover.

   - Step 1: Check if $S$ is closed. The point 0 is a limit point of $S$, but $0 \notin S$. Therefore, $S$ is not closed.
   - Step 2: Since $S$ is not closed, it cannot be compact.
   - Step 3: Construct a cover. Let $U_n = (\frac{1}{n}, 2)$ for $n = 1, 2, 3, \ldots$. The union is $(0, 2)$, which covers $(0, 1]$. Any finite subcover will have a maximum $n$, say $N$, and will only cover $(\frac{1}{N}, 2)$, missing points in $(0, \frac{1}{N}]$.

   **Answer:** No, it is not compact.

3. Is the set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ compact? Hint: Consider the limit points of the set.

   - Step 1: Identify the limit points. The sequence $\frac{1}{n}$ converges to 0, so 0 is a limit point of $S$.
   - Step 2: Check if $S$ is closed. The limit point 0 is not in the set $S$ (since $\frac{1}{n}$ is never 0 for any natural number $n$). Thus, $S$ is not closed.
   - Step 3: Since $S$ is not closed, it is not compact.

   **Answer:** No, it is not compact.

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