

Interior, closure, boundary, limit points

Lesson 46 · Module 4 — Topology of the Real Line · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Find the interior, closure, and boundary of the set $S = [-1, 0] \cup (0, 1)$.
2. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Find the set of limit points and the closure of S .
3. Prove that if S is an open set, then $\partial S = \text{cl}(S) \setminus S$.

Solutions

1. Find the interior, closure, and boundary of the set $S = [-1, 0] \cup (0, 1)$.

- The set S is the union of all x from -1 to 0 inclusive and all x from 0 to 1 exclusive, which simplifies to the interval $[-1, 1)$.
- The interior consists of all points x such that $(-1 < x < 1)$, so $\text{int}(S) = (-1, 1)$. Note that $x = 0$ is an interior point because the interval $(-0.5, 0.5)$ is a subset of $[-1, 1)$.
- The limit points are all x in $[-1, 1]$, so the closure is $\text{cl}(S) = [-1, 1]$.
- The boundary is the closure minus the interior: $[-1, 1] \setminus (-1, 1) = \{-1, 1\}$.

Answer: $\text{int}(S) = (-1, 1)$, $\text{cl}(S) = [-1, 1]$, $\partial S = \{-1, 1\}$

2. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Find the set of limit points and the closure of S .

- For any n , the point $1/n$ is isolated because we can pick a small enough ϵ to exclude other $1/m$.
- However, as n goes to infinity, $1/n$ approaches 0 .
- Every neighborhood of 0 contains points of the form $1/n$ for sufficiently large n .
- Thus, the only limit point is 0 .
- The closure is $S \cup \{0\}$.

Answer: $\text{Limit points} = \{0\}$, $\text{cl}(S) = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

3. Prove that if S is an open set, then $\partial S = \text{cl}(S) \setminus S$.

- By definition, $\partial S = \text{cl}(S) \setminus \text{int}(S)$.
- Since S is open, every point in S is an interior point, so $S \subset \text{int}(S)$.
- Also, the interior is always a subset of the set, so $\text{int}(S) \subset S$.
- Therefore, for an open set, $\text{int}(S) = S$.
- Substituting this into the boundary formula gives $\partial S = \text{cl}(S) \setminus S$.

Answer: $\partial S = \text{cl}(S) \setminus S$