

Module 3 Review and Self-Test

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Practice problems

1. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges or diverges. Justify your answer using a rigorous test.
2. Consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Does this series converge absolutely, conditionally, or diverge? Explain.
3. Compute the first three terms c_0, c_1, c_2 of the Cauchy product of the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ with itself.

Solutions

1. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges or diverges. Justify your answer using a rigorous test.
 - Observe that for $n \geq 1$, $n^3 + 1 > n^3$.
 - Therefore, $\sqrt{n^3+1} > \sqrt{n^3} = n^{3/2}$.
 - This implies $\frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$.
 - The series $\sum \frac{1}{n^{3/2}}$ is a p-series with $p = 1.5 > 1$, so it converges.
 - By the Comparison Test, the original series also converges.

Answer: Converges

2. Consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Does this series converge absolutely, conditionally, or diverge? Explain.
 - Check absolute convergence: $\sum \left| \frac{(-1)^{n+1}}{n} \right| = \sum \frac{1}{n}$.
 - The series $\sum \frac{1}{n}$ is the harmonic series, which is known to diverge.
 - Check convergence of the original series: the terms $a_n = \frac{1}{n}$ are positive, decreasing, and $\lim_{n \rightarrow \infty} a_n = 0$.
 - By the Alternating Series Theorem, the series converges.
 - Since it converges but not absolutely, it converges conditionally.

Answer: Conditionally convergent

3. Compute the first three terms c_0, c_1, c_2 of the Cauchy product of the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ with itself.
 - The terms are $a_n = \frac{1}{3^n}$ and $b_n = \frac{1}{3^n}$.
 - For $n = 0$: $c_0 = a_0b_0 = 1 \cdot 1 = 1$.
 - For $n = 1$: $c_1 = a_0b_1 + a_1b_0 = 1 \cdot \frac{1}{3} + \frac{1}{3} \cdot 1 = \frac{2}{3}$.
 - For $n = 2$: $c_2 = a_0b_2 + a_1b_1 + a_2b_0 = 1 \cdot \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{9} \cdot 1 = \frac{3}{9} = \frac{1}{3}$.

Answer: $c_0 = 1$, $c_1 = 2/3$, $c_2 = 1/3$