Products of series; Cauchy products

Lesson 42 · Module 3 — Series, Rigorously · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Find the Cauchy product of the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ and $\sum_{n=0}^{\infty} \frac{1}{3^n}$. Show that the resulting series converges to the product of the individual sums.
2. Let $a_n = \frac{1}{n!}$ and $b_n = \frac{1}{n!}$. Find the general term c_n of their Cauchy product and identify the resulting series.
3. Explain why the Cauchy product of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ with itself does not converge to $(\ln 2)^2$.

Solutions

1. Find the Cauchy product of the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ and $\sum_{n=0}^{\infty} \frac{1}{3^n}$. Show that the resulting series converges to the product of the individual sums.

   - The individual sums are geometric series: $\sum(1/3)^n = 1/(1-1/3) = 3/2$. The product of the sums is $(3/2)\cdot(3/2) = 9/4$.
   - Compute $c_n = \sum_{k=0}^{n}(1/3)^k(1/3)^{n-k} = \sum_{k=0}^{n}(1/3)^n = (n+1)/3^n$.
   - Sum the resulting series: $\sum_{n=0}^{\infty} \frac{n+1}{3^n} = \sum \frac{n}{3^n} + \sum \frac{1}{3^n}$.
   - The geometric part is $3/2$. For the arithmetico-geometric part, we use the known result $\sum_{n=0}^{\infty} x^n = 1/(1-x)$. Differentiating term-by-term (Calculus L114) gives $\sum nx^{n-1} = 1/(1-x)^2$. Multiplying by x gives $\sum nx^n = x/(1-x)^2$. For $x = 1/3$, this is $(1/3)/(4/9) = 3/4$.
   - Total sum is $3/4 + 3/2 = 9/4$.

   Answer: $\sum c_n = 9/4$

2. Let $a_n = \frac{1}{n!}$ and $b_n = \frac{1}{n!}$. Find the general term c_n of their Cauchy product and identify the resulting series.

   - The terms are $c_n = \sum_{k=0}^{n} \frac{1}{k!} \frac{1}{(n-k)!}$.
   - Multiply and divide by $n!$: $c_n = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}$.
   - Recognize the binomial coefficient: $c_n = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}$.
   - Use the identity $\sum \binom{n}{k} = 2^n$: $c_n = \frac{2^n}{n!}$.
   - The resulting series is $\sum \frac{2^n}{n!}$, which is the Taylor series for e^2.

   Answer: $c_n = \frac{2^n}{n!}$

3. Explain why the Cauchy product of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ with itself does not converge to $(\ln 2)^2$.

   - The series $\sum \frac{(-1)^n}{n+1}$ converges conditionally, not absolutely.
   - Mertens' Theorem requires at least one series to converge absolutely to guarantee the product of sums.
   - As shown in the lesson, the terms c_n of the Cauchy product of the alternating harmonic series behave like $2\ln n/n$ for large n.
   - The series $\sum 2\ln n/n$ diverges by comparison with the harmonic series $\sum 1/n$.
   - Since the Cauchy product diverges, it cannot converge to the finite value $(\ln 2)^2$.

   Answer: The series is conditionally convergent and the resulting Cauchy product diverges.
