The rearrangement theorem II — any sum you like

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Practice problems

1. Describe the greedy algorithm process to rearrange the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ so that it converges to 1. Which term is the first to be added, and when does the first switch to negative terms occur?

2. If a series $\sum a_n$ converges absolutely to S, does there exist a rearrangement σ such that $\sum a_{\sigma(n)} = S + 1$? Explain why or why not.

3. Suppose $\sum a_n$ is conditionally convergent. Prove that we can rearrange it to diverge to $-\infty$.

Solutions

1. Describe the greedy algorithm process to rearrange the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ so that it converges to 1. Which term is the first to be added, and when does the first switch to negative terms occur?

    - The first term is 1.
    - Since the current sum is 1, and our target is 1, the algorithm immediately switches to negative terms to avoid exceeding the target.
    - We add negative terms $-\frac{1}{2}, -\frac{1}{4}, \ldots$ until the sum is less than 1.
    - Then we add positive terms $\frac{1}{3}, \frac{1}{5}, \ldots$ until the sum exceeds 1 again.

    Answer: Start with 1, then add negative terms until the sum is < 1, then positive terms until the sum is > 1.

2. If a series $\sum a_n$ converges absolutely to S, does there exist a rearrangement σ such that $\sum a_{\sigma(n)} = S + 1$? Explain why or why not.

    - No, such a rearrangement does not exist.
    - A fundamental property of absolutely convergent series is that every rearrangement converges to the same sum.
    - Therefore, for any permutation σ, $\sum a_{\sigma(n)} = S$.
    - Since $S \neq S + 1$, the target sum is impossible.

    Answer: No, because absolutely convergent series are invariant under rearrangement.

3. Suppose $\sum a_n$ is conditionally convergent. Prove that we can rearrange it to diverge to $-\infty$.

    - Use the same strategy as the divergence to $+\infty$, but mirrored.
    - Add negative terms until the partial sum is less than -1.
    - Add exactly one positive term to ensure all terms are used.
    - Add negative terms until the partial sum is less than -2, and so on.
    - Since $\sum a_n^- = -\infty$, we can always push the sum below any negative integer $-k$.
    - Since $a_n \to 0$, the single positive terms cannot stop the divergence.

    Answer: By greedily adding negative terms to pass $-1, -2, -3, \ldots$ and adding single positive terms to maintain the bijection.
