The rearrangement theorem I — conditional convergence's dark secret

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Practice problems

1. For the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$, determine if the series converges absolutely or conditionally, and then find the behavior of $\sum a_n^+$ and $\sum a_n^-$.
2. Suppose $\sum a_n$ is a series such that $\sum a_n^+$ converges. Prove that $\sum a_n$ converges if and only if $\sum a_n^-$ converges.
3. Can a series be conditionally convergent if $\sum a_n^+$ converges? Explain why or why not.

Solutions

1. For the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$, determine if the series converges absolutely or conditionally, and then find the behavior of $\sum a_n^+$ and $\sum a_n^-$.

   - The series is alternating with $a_n = \frac{1}{n\ln(n)}$. Since a_n decreases to 0, the series converges by the Alternating Series Theorem.
   - The absolute series is $\sum \frac{1}{n\ln(n)}$. Using the integral test, $\int_2^{\infty} \frac{1}{x\ln(x)} dx = [\ln(\ln(x))]_2^{\infty} = \infty$. Thus, it diverges.
   - Since the series converges but not absolutely, it is conditionally convergent.
   - By today's theorem, $\sum a_n^+ = \infty$ and $\sum a_n^- = -\infty$.

   Answer: Conditionally convergent; $\sum a_n^+ = \infty$ and $\sum a_n^- = -\infty$.

2. Suppose $\sum a_n$ is a series such that $\sum a_n^+$ converges. Prove that $\sum a_n$ converges if and only if $\sum a_n^-$ converges.

   - We know $a_n = a_n^+ + a_n^-$.
   - If $\sum a_n^-$ converges, then $\sum a_n$ is the sum of two convergent series, so it must converge.
   - If $\sum a_n$ converges, then $\sum a_n^- = \sum a_n - \sum a_n^+$.
   - Since the difference of two convergent series is convergent, $\sum a_n^-$ must converge.

   Answer: The statement is true by the linearity of convergent series.

3. Can a series be conditionally convergent if $\sum a_n^+$ converges? Explain why or why not.

   - If $\sum a_n^+$ converges, then for the series to be conditionally convergent, $\sum |a_n|$ must diverge.
   - We know $\sum |a_n| = \sum a_n^+ - \sum a_n^-$.
   - If $\sum a_n^+$ converges and $\sum |a_n|$ diverges, then $\sum a_n^-$ must diverge to $-\infty$.
   - However, if $\sum a_n^+$ converges and $\sum a_n^-$ diverges, then the total sum $\sum a_n = \sum a_n^+ + \sum a_n^-$ must also diverge to $-\infty$.
   - This contradicts the definition of conditional convergence, which requires the total sum to converge.

   Answer: No, because if $\sum a_n^+$ converges, the total series can only converge if $\sum a_n^-$ also converges, which would imply absolute convergence.
