Alternating series theorem — proved with error bound

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Practice problems

1. Determine if the series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$ converges. Justify your answer using the Alternating Series Theorem.
2. For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$, how many terms n are needed to ensure the partial sum S_n is within 0.01 of the actual sum S?
3. Does the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n}$ converge? Explain why or why not.

Solutions

1. Determine if the series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$ converges. Justify your answer using the Alternating Series Theorem.

    - Check if terms alternate: The factor $(-1)^{n+1}$ ensures the signs flip every term.
    - Check if absolute values $a_n = \frac{1}{\ln(n+1)}$ are non-increasing: Since $\ln(x)$ is an increasing function, $\ln(n+2) > \ln(n+1)$, which means $\frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)}$.
    - Check the limit: $\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$ because the logarithm grows without bound.
    - Since all three conditions are met, the series converges.

    Answer: Converges

2. For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$, how many terms n are needed to ensure the partial sum S_n is within 0.01 of the actual sum S?

    - The error bound is $|S - S_n| \leq a_{n+1}$.
    - We need $a_{n+1} \leq 0.01$, where $a_{n+1} = \frac{1}{(n+1)^2}$.
    - Solve $\frac{1}{(n+1)^2} \leq \frac{1}{100}$.
    - This implies $(n+1)^2 \geq 100$, so $n + 1 \geq 10$.
    - Therefore, $n \geq 9$.

    Answer: n = 9

3. Does the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n}$ converge? Explain why or why not.

    - Check the limit of the absolute terms: $a_n = \frac{n+1}{n} = 1 + \frac{1}{n}$.
    - Compute the limit: $\lim_{n\to\infty}(1 + \frac{1}{n}) = 1$.
    - The Alternating Series Theorem requires the limit to be 0.
    - Since the limit is 1, the terms of the series do not approach 0.
    - By the Divergence Test, the series diverges.

    Answer: Diverges
