Absolute convergence; ratio and root tests proved

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Practice problems

1. Determine if the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges using the Ratio Test.
2. Use the Root Test to determine the convergence of $\sum_{n=1}^{\infty} \left(\frac{3n+1}{2n-1}\right)^n$.
3. Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely using the Ratio Test.

Solutions

1. Determine if the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges using the Ratio Test.

   - Identify $a_n = \frac{n!}{n^n}$ and $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$.
   - Compute the ratio: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)n!}{(n+1)(n+1)^n} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n}$.
   - Rewrite the ratio as $\left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n}$.
   - Take the limit as $n \to \infty$: $\lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e}$.
   - Since $e \approx 2.718$, $1/e < 1$.
   - By the Ratio Test, the series converges absolutely.

   Answer: Converges absolutely

2. Use the Root Test to determine the convergence of $\sum_{n=1}^{\infty} \left(\frac{3n+1}{2n-1}\right)^n$.

   - Identify $a_n = \left(\frac{3n+1}{2n-1}\right)^n$.
   - Apply the n-th root: $\sqrt[n]{|a_n|} = \frac{3n+1}{2n-1}$.
   - Take the limit as $n \to \infty$: $\lim_{n\to\infty} \frac{3n+1}{2n-1} = \frac{3}{2}$.
   - Since $3/2 > 1$, the Root Test implies the series diverges.

   Answer: Diverges

3. Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely using the Ratio Test.

   - Identify $a_n = \frac{(-1)^n}{n^2}$.
   - Apply the Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$.
   - The Ratio Test is inconclusive because $L = 1$.
   - Check for absolute convergence by looking at $\sum |a_n| = \sum \frac{1}{n^2}$.
   - This is a p-series with $p = 2$, which converges (Lesson 105).
   - Therefore, the series converges absolutely.

   Answer: Converges absolutely
