Comparison and condensation — harmonic divergence, third proof

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Practice problems

1. Use the Comparison Test to determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges or diverges.
2. Use the Cauchy Condensation Test to determine the convergence of $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$.
3. Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ using the Comparison Test.

Solutions

1. Use the Comparison Test to determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges or diverges.

   - Observe that for $n \geq 1$, $n^2 + n > n^2$.
   - Therefore, $\frac{1}{n^2+n} < \frac{1}{n^2}$.
   - We know that $\sum \frac{1}{n^2}$ is a convergent p-series with $p = 2$.
   - By the Comparison Test, since our series is term-by-term smaller than a convergent series, it must also converge.

   Answer: Converges

2. Use the Cauchy Condensation Test to determine the convergence of $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$.

   - The terms $a_n = \frac{1}{n(\ln n)^3}$ are positive and non-increasing.
   - Apply condensation: $2^k a_{2^k} = 2^k \cdot \frac{1}{2^k(\ln 2^k)^3} = \frac{1}{(k \ln 2)^3}$.
   - This simplifies to $\frac{1}{(\ln 2)^3} \cdot \frac{1}{k^3}$.
   - The series $\sum \frac{1}{k^3}$ is a convergent p-series with $p = 3$.
   - Since the condensed series converges, the original series converges.

   Answer: Converges

3. Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ using the Comparison Test.

   - Observe that for $n \geq 1$, $n + \sqrt{n} \leq n + n = 2n$.
   - Therefore, $\frac{1}{n+\sqrt{n}} \geq \frac{1}{2n}$.
   - The series $\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ is a multiple of the harmonic series, which diverges.
   - By the Comparison Test, since our series is term-by-term larger than a divergent series, it must also diverge.

   Answer: Diverges
