The Cauchy criterion for series; the divergence test justified

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Practice problems

1. Use the Cauchy criterion to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Hint: Use the inequality $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ for $k \geq 2$.
2. Suppose a series $\sum a_n$ converges. Prove that for any $\epsilon > 0$, there exists N such that $|a_n| < \epsilon$ for all $n > N$.
3. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ satisfy the Cauchy criterion? Explain your reasoning based on the definition.

Solutions

1. Use the Cauchy criterion to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Hint: Use the inequality $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ for $k \geq 2$.

   - Consider the block sum $\sum_{k=n+1}^{m} \frac{1}{k^2}$.
   - Apply the hint: $\sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} (\frac{1}{k-1} - \frac{1}{k})$.
   - This is a telescoping sum: $(\frac{1}{n} - \frac{1}{n+1}) + (\frac{1}{n+1} - \frac{1}{n+2}) + \cdots + (\frac{1}{m-1} - \frac{1}{m})$.
   - The sum simplifies to $\frac{1}{n} - \frac{1}{m}$.
   - Since $\frac{1}{n} - \frac{1}{m} < \frac{1}{n}$, we can make this less than ϵ by choosing $N > \frac{1}{\epsilon}$.
   - Thus, the Cauchy criterion is satisfied.

   Answer: The series converges because the block sum is bounded by $\frac{1}{n}$, which can be made arbitrarily small.

2. Suppose a series $\sum a_n$ converges. Prove that for any $\epsilon > 0$, there exists N such that $|a_n| < \epsilon$ for all $n > N$.

   - Since the series converges, it satisfies the Cauchy criterion.
   - For any $\epsilon > 0$, there exists N such that for all $m > n \geq N$, $|\sum_{k=n+1}^{m} a_k| < \epsilon$.
   - Choose $m = n + 1$.
   - The sum becomes $|\sum_{k=n+1}^{n+1} a_k| = |a_{n+1}|$.
   - Therefore, $|a_{n+1}| < \epsilon$ for all $n \geq N$, which means $|a_k| < \epsilon$ for all $k > N$.

   Answer: The result follows directly from the Cauchy criterion by setting $m = n + 1$.

3. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ satisfy the Cauchy criterion? Explain your reasoning based on the definition.

   - The series is the alternating harmonic series.
   - We know from the Alternating Series Test (Calculus, Lesson 107) that this series converges.
   - By the theorem proved today, a series converges if and only if it satisfies the Cauchy criterion.
   - Therefore, the alternating harmonic series must satisfy the Cauchy criterion.
   - Specifically, the block sum $|S_m - S_n|$ for an alternating series with decreasing terms is bounded by the magnitude of the first omitted term, $|a_{n+1}| = \frac{1}{n+1}$.
   - Since $\frac{1}{n+1} \to 0$, the Cauchy criterion is satisfied.

   Answer: Yes, it satisfies the Cauchy criterion because the series converges.
