Module 2 Review and Self-Test

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Practice problems

1. Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$. Prove that the sequence converges and find its limit.
2. Determine the $\liminf$ and $\limsup$ of the sequence $a_n = \sin\left(\frac{n\pi}{2}\right) + \frac{1}{n}$.
3. Prove that the sequence $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is Cauchy.

Solutions

1. Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$. Prove that the sequence converges and find its limit.

    - Use induction to show $x_n < 2$. For $n = 1$, $1 < 2$. If $x_n < 2$, then $x_{n+1} = \sqrt{2 + x_n} < \sqrt{2 + 2} = 2$.
    - Show the sequence is increasing. $x_{n+1}^2 - x_n^2 = 2 + x_n - x_n^2 = (2 - x_n)(1 + x_n)$. Since $x_n < 2$ and $x_n > 0$, this is positive, so $x_{n+1} > x_n$.
    - By the Monotone Convergence Theorem, the sequence converges to a limit L.
    - Solve $L = \sqrt{2 + L}$, which gives $L^2 - L - 2 = 0$. The roots are 2 and -1. Since $x_n > 0$, $L = 2$.

    Answer: $L = 2$

2. Determine the lim inf and lim sup of the sequence $a_n = \sin\left(\frac{n\pi}{2}\right) + \frac{1}{n}$.

    - The term $\sin(n\pi/2)$ takes values $1, 0, -1, 0$ periodically.
    - The sequence a_n takes values $1 + 1/1, 0 + 1/2, -1 + 1/3, 0 + 1/4, 1 + 1/5, \ldots$
    - The subsequence of terms where $\sin(n\pi/2) = 1$ converges to 1.
    - The subsequence of terms where $\sin(n\pi/2) = -1$ converges to -1.
    - The subsequence of terms where $\sin(n\pi/2) = 0$ converges to 0.
    - The highest limit point is 1 and the lowest is -1.

    Answer: $\liminf a_n = -1, \limsup a_n = 1$

3. Prove that the sequence $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is Cauchy.

    - Consider $m > n$. The difference is $|a_m - a_n| = \sum_{k=n+1}^{m} \frac{1}{k^2}$.
    - Use the inequality $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.
    - The sum is a telescoping sum: $\sum_{k=n+1}^{m} (\frac{1}{k-1} - \frac{1}{k}) = \frac{1}{n} - \frac{1}{m}$.
    - Thus $|a_m - a_n| < \frac{1}{n}$.
    - For any $\epsilon > 0$, choose $N > 1/\epsilon$. Then for $m, n \geq N$, $|a_m - a_n| < 1/N < \epsilon$.
    - Therefore, the sequence is Cauchy.

    Answer: The sequence is Cauchy because $|a_m - a_n| < 1/n$ for $m > n$.
