Recursive sequences clinic — root 2 by iteration

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Practice problems

1. Consider the sequence $x_{n+1} = \frac{1}{2}(x_n + \frac{3}{x_n})$ with $x_1 = 2$. Prove that the sequence converges and find its limit.
2. Analyze the convergence of $x_{n+1} = \sqrt{6 + x_n}$ with $x_1 = \sqrt{6}$. Find the limit if it exists.
3. Does the sequence $x_{n+1} = 3x_n - 2$ with $x_1 = 1$ converge? Justify your answer.

Solutions

1. Consider the sequence $x_{n+1} = \frac{1}{2}(x_n + \frac{3}{x_n})$ with $x_1 = 2$. Prove that the sequence converges and find its limit.

   - Step 1: Show $x_n \geq \sqrt{3}$ for $n \geq 2$. Consider $(x_n - \sqrt{3}/x_n)^2 \geq 0$, which implies $x_n^2 + 3/x_n^2 \geq 2\sqrt{3}$. Alternatively, use the fact that $x_{n+1}^2 - 3 = \frac{(x_n^2-3)^2}{4x_n^2} \geq 0$, so $x_n \geq \sqrt{3}$ for $n \geq 2$.
   - Step 2: Show the sequence is decreasing for $n \geq 2$ by calculating $x_{n+1} - x_n = \frac{3-x_n^2}{2x_n}$. Since $x_n^2 \geq 3$, the difference is ≤ 0.
   - Step 3: By the Monotone Convergence Theorem, the limit L exists.
   - Step 4: Solve $L = \frac{1}{2}(L+3/L)$, which gives $2L = L+3/L$, so $L^2 = 3$. Since $x_n > 0$, $L = \sqrt{3}$.

   Answer: $\sqrt{3}$

2. Analyze the convergence of $x_{n+1} = \sqrt{6 + x_n}$ with $x_1 = \sqrt{6}$. Find the limit if it exists.

   - Step 1: Use induction to show $x_n < 3$. $x_1 = \sqrt{6} < 3$. If $x_n < 3$, then $x_{n+1} = \sqrt{6 + x_n} < \sqrt{6+3} = 3$.
   - Step 2: Show the sequence is increasing. $x_{n+1}^2 - x_n^2 = 6 + x_n - x_n^2 = (3 - x_n)(2 + x_n)$. Since $x_n < 3$ and $x_n > 0$, the difference is positive.
   - Step 3: By MCT, the limit L exists.
   - Step 4: Solve $L = \sqrt{6 + L} \implies L^2 - L - 6 = 0 \implies (L-3)(L+2) = 0$. Since $x_n > 0$, $L = 3$.

   Answer: 3

3. Does the sequence $x_{n+1} = 3x_n - 2$ with $x_1 = 1$ converge? Justify your answer.

   - Step 1: Calculate the first few terms: $x_1 = 1$, $x_2 = 3(1) - 2 = 1$, $x_3 = 3(1) - 2 = 1$.
   - Step 2: Observe that $x_n = 1$ for all n.
   - Step 3: A constant sequence is both monotone and bounded.
   - Step 4: The limit is simply the constant value 1.

   Answer: Yes, it converges to 1.
