limsup and liminf — taming non-convergence

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Practice problems

1. Find the $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ for the sequence $a_n = \frac{1+\sin(n)}{2}$.
2. Consider the sequence $a_n = \frac{(-1)^n}{n} + 2$. Determine if the sequence converges by calculating its $\limsup$ and $\liminf$.
3. Let a_n be a bounded sequence. If $\limsup_{n\to\infty} a_n = 5$ and $\liminf_{n\to\infty} a_n = 2$, can the sequence converge? Explain why or why not.

Solutions

1. Find the $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ for the sequence $a_n = \frac{1+\sin(n)}{2}$.

   - We know that $-1 \leq \sin(n) \leq 1$ for all n.
   - Adding 1 to all sides gives $0 \leq 1 + \sin(n) \leq 2$.
   - Dividing by 2 gives $0 \leq \frac{1+\sin(n)}{2} \leq 1$.
   - Since $\sin(n)$ gets arbitrarily close to 1 and -1 infinitely often, the supremum of the tail approaches 1 and the infimum of the tail approaches 0.

   Answer: \limsup a_n = 1, \liminf a_n = 0

2. Consider the sequence $a_n = \frac{(-1)^n}{n} + 2$. Determine if the sequence converges by calculating its lim sup and lim inf.

   - The term $\frac{(-1)^n}{n}$ converges to 0 as $n \to \infty$.
   - Therefore, a_n converges to $0 + 2 = 2$.
   - For a convergent sequence, $\limsup a_n = \liminf a_n = L$.
   - Thus, $\limsup a_n = 2$ and $\liminf a_n = 2$.

   Answer: \limsup a_n = \liminf a_n = 2, \text so it converges to 2

3. Let a_n be a bounded sequence. If $\limsup_{n\to\infty} a_n = 5$ and $\liminf_{n\to\infty} a_n = 2$, can the sequence converge? Explain why or why not.

   - A bounded sequence converges if and only if its lim sup and lim inf are equal.
   - In this case, $\limsup a_n = 5$ and $\liminf a_n = 2$.
   - Since $5 \neq 2$, the condition for convergence is not met.
   - The sequence must oscillate and therefore diverges.

   Answer: \textNo, because \limsup a_n \neq \liminf a_n