Cauchy sequences II — completeness, three equivalent faces

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Practice problems

1. Use the Cauchy criterion to prove that the sequence $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ converges. Hint: use the inequality $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ for $k \geq 2$.
2. Prove that the sequence $a_n = \ln(n)$ is not Cauchy.
3. Suppose a sequence a_n is Cauchy and has a subsequence a_{n_k} that converges to L. Prove that a_n converges to L.

Solutions

1. Use the Cauchy criterion to prove that the sequence $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ converges. Hint: use the inequality $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ for $k \geq 2$.

   - Consider $m > n$. The distance $|a_m - a_n| = \sum_{k=n+1}^{m} \frac{1}{k^2}$.
   - Using the hint, $\sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} (\frac{1}{k-1} - \frac{1}{k})$.
   - This is a telescoping sum: $(\frac{1}{n} - \frac{1}{n+1}) + (\frac{1}{n+1} - \frac{1}{n+2}) + \cdots + (\frac{1}{m-1} - \frac{1}{m}) = \frac{1}{n} - \frac{1}{m}$.
   - Since $\frac{1}{n} - \frac{1}{m} < \frac{1}{n}$, we can make this less than ϵ by choosing $N > \frac{1}{\epsilon}$.
   - Thus, the sequence is Cauchy and therefore converges by completeness.

   Answer: The sequence is Cauchy because $|a_m - a_n| < \frac{1}{n}$, so it converges.

2. Prove that the sequence $a_n = \ln(n)$ is not Cauchy.

   - To show it is not Cauchy, we need to find $\epsilon > 0$ such that for any N, there exist $m, n \geq N$ with $|a_m - a_n| \geq \epsilon$.
   - Let $\epsilon = \ln(2)$.
   - For any N, choose $n = N$ and $m = 2N$.
   - Then $|a_m - a_n| = |\ln(2N) - \ln(N)| = |\ln(\frac{2N}{N})| = \ln(2)$.
   - Since we found a constant distance $\ln(2)$ that persists for any N, the sequence is not Cauchy.

   Answer: The sequence is not Cauchy because $|a_{2n} - a_n| = \ln(2)$ for all n.

3. Suppose a sequence a_n is Cauchy and has a subsequence a_{n_k} that converges to L. Prove that a_n converges to L.

   - Let $\epsilon > 0$. Since a_n is Cauchy, there exists N such that $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N$.
   - Since $a_{n_k} \to L$, there exists K such that for $k \geq K$, $|a_{n_k} - L| < \epsilon/2$.
   - Pick k large enough such that $k \geq K$ and $n_k \geq N$.
   - For any $n \geq N$, use the triangle inequality: $|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L|$.
   - The first term is $< \epsilon/2$ because $n, n_k \geq N$. The second term is $< \epsilon/2$ because $k \geq K$.
   - Thus $|a_n - L| < \epsilon$ for all $n \geq N$.

   Answer: The sequence converges to L by the triangle inequality and the Cauchy property.
