Cauchy sequences I — convergence without knowing the limit

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Practice problems

1. Prove that the sequence $a_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$ is a Cauchy sequence using the definition.
2. Show that the sequence $a_n = \sqrt{n}$ is not a Cauchy sequence.
3. Suppose a sequence a_n satisfies $|a_{n+1} - a_n| < \frac{1}{2^n}$ for all $n \geq 1$. Prove that a_n is a Cauchy sequence.

Solutions

1. Prove that the sequence $a_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$ is a Cauchy sequence using the definition.

   - Step 1: Consider $n > m$. The difference is $|a_n - a_m| = \sum_{k=m+1}^{n} \frac{1}{k(k+1)}$.
   - Step 2: Use partial fractions: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.
   - Step 3: The sum telescopes: $\sum_{k=m+1}^{n}(\frac{1}{k} - \frac{1}{k+1}) = \frac{1}{m+1} - \frac{1}{n+1}$.
   - Step 4: Bound the result: $|a_n - a_m| < \frac{1}{m+1} < \frac{1}{m}$.
   - Step 5: For any $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$. Then for $n, m \geq N$, $|a_n - a_m| < \frac{1}{N} < \epsilon$.

   Answer: The sequence is Cauchy because for any $\epsilon > 0$, choosing $N > 1/\epsilon$ ensures $|a_n - a_m| < \epsilon$ for all $n, m \geq N$.

2. Show that the sequence $a_n = \sqrt{n}$ is not a Cauchy sequence.

   - Step 1: To show it is not Cauchy, we must find an $\epsilon > 0$ such that for any N, there exist $n, m \geq N$ with $|a_n - a_m| \geq \epsilon$.
   - Step 2: Let $\epsilon = 1$.
   - Step 3: For any N, choose $m = N$ and n such that $\sqrt{n} \geq \sqrt{N} + 1$. This is possible by picking $n \geq (\sqrt{N} + 1)^2$.
   - Step 4: Then $|a_n - a_m| = \sqrt{n} - \sqrt{N} \geq 1 = \epsilon$.

   Answer: The sequence is not Cauchy because the distance between terms can be made arbitrarily large.

3. Suppose a sequence a_n satisfies $|a_{n+1} - a_n| < \frac{1}{2^n}$ for all $n \geq 1$. Prove that a_n is a Cauchy sequence.

   - Step 1: Assume $n > m$. Use the triangle inequality to write $|a_n - a_m| \leq \sum_{k=m}^{n-1} |a_{k+1} - a_k|$.
   - Step 2: Substitute the given bound: $|a_n - a_m| < \sum_{k=m}^{n-1} \frac{1}{2^k}$.
   - Step 3: This is a geometric sum. The sum is $\frac{1}{2^m}(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}) < \frac{1}{2^m} \cdot 2 = \frac{1}{2^{m-1}}$.
   - Step 4: For any $\epsilon > 0$, we can choose N such that $\frac{1}{2^{N-1}} < \epsilon$.
   - Step 5: Then for all $n, m \geq N$, $|a_n - a_m| < \frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}} < \epsilon$.

   Answer: The sequence is Cauchy because the sum of the gaps between consecutive terms is bounded by a convergent geometric series.
