Bolzano-Weierstrass — the compactness engine

Lesson 29 · Module 2 — Sequences · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Consider the sequence $a_n = \frac{(-1)^n}{n} + 5$. Is this sequence bounded? Does it have a convergent subsequence? If so, what is the limit?
2. Let $a_n = \sin(\frac{n\pi}{2})$. Find all possible subsequential limits of this sequence.
3. Prove that the sequence $a_n = (-1)^n(1 + \frac{1}{n})$ has exactly two subsequential limits and find them.

Solutions

1. Consider the sequence $a_n = \frac{(-1)^n}{n} + 5$. Is this sequence bounded? Does it have a convergent subsequence? If so, what is the limit?

   - Step 1: Check boundedness. Since $|\frac{(-1)^n}{n}| \leq 1$, we have $4 \leq a_n \leq 6$. The sequence is bounded.
   - Step 2: Apply Bolzano-Weierstrass. Since it is bounded, a convergent subsequence must exist.
   - Step 3: Evaluate the limit. As $n \to \infty$, $\frac{(-1)^n}{n} \to 0$. Thus, the original sequence $a_n \to 5$.
   - Step 4: Since the original sequence converges to 5, every subsequence also converges to 5.

   Answer: The sequence is bounded and every convergent subsequence converges to 5.

2. Let $a_n = \sin(\frac{n\pi}{2})$. Find all possible subsequential limits of this sequence.

   - Step 1: List the first few terms: $\sin(\pi/2) = 1, \sin(\pi) = 0, \sin(3\pi/2) = -1, \sin(2\pi) = 0$.
   - Step 2: The sequence repeats the pattern $1, 0, -1, 0$.
   - Step 3: The subsequence a_{4k+1} is constant 1, so it converges to 1.
   - Step 4: The subsequence a_{4k+3} is constant -1, so it converges to -1.
   - Step 5: The subsequence a_{2k} is constant 0, so it converges to 0.

   Answer: The subsequential limits are $\{-1, 0, 1\}$.

3. Prove that the sequence $a_n = (-1)^n(1 + \frac{1}{n})$ has exactly two subsequential limits and find them.

   - Step 1: Split the sequence into even and odd indices.
   - Step 2: For even $n = 2k$, $a_{2k} = (-1)^{2k}(1 + \frac{1}{2k}) = 1 + \frac{1}{2k}$. As $k \to \infty$, $a_{2k} \to 1$.
   - Step 3: For odd $n = 2k-1$, $a_{2k-1} = (-1)^{2k-1}(1 + \frac{1}{2k-1}) = -(1 + \frac{1}{2k-1})$. As $k \to \infty$, $a_{2k-1} \to -1$.
   - Step 4: Any other subsequence must be a combination of these terms. Since any subsequence of a convergent sequence converges to the same limit, any convergent subsequence must converge to either 1 or -1.

   Answer: The subsequential limits are 1 and -1.
