Subsequences; sequential characterisations

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Practice problems

1. Determine if the sequence $a_n = \begin{cases} 1 + \frac{1}{n} & n \text{ even} \\ 2 + \frac{1}{n} & n \text{ odd} \end{cases}$ converges or diverges.
2. Let $a_n = \frac{(-1)^n}{n}$. Prove that a_n converges to 0 using subsequences.
3. Does the sequence $a_n = \sin(\frac{n\pi}{4})$ converge? Justify your answer using subsequential limits.

Solutions

1. Determine if the sequence $a_n = \begin{cases} 1 + \frac{1}{n} & n \text{ even} \\ 2 + \frac{1}{n} & n \text{ odd} \end{cases}$ converges or diverges.

   - Consider the subsequence of even terms $a_{2k} = 1 + \frac{1}{2k}$. As $k \to \infty$, $a_{2k} \to 1$.
   - Consider the subsequence of odd terms $a_{2k-1} = 2 + \frac{1}{2k-1}$. As $k \to \infty$, $a_{2k-1} \to 2$.
   - Since the two subsequences converge to different limits ($1 \neq 2$), the original sequence diverges.

   Answer: Diverges

2. Let $a_n = \frac{(-1)^n}{n}$. Prove that a_n converges to 0 using subsequences.

   - The even subsequence is $a_{2k} = \frac{1}{2k}$, which converges to 0.
   - The odd subsequence is $a_{2k-1} = \frac{-1}{2k-1}$, which converges to 0.
   - Since every term of the sequence is either in the even or odd subsequence, and both converge to 0, the entire sequence a_n converges to 0.

   Answer: \lim_n \to \infty a_n = 0

3. Does the sequence $a_n = \sin(\frac{n\pi}{4})$ converge? Justify your answer using subsequential limits.

   - Pick $n_k = 8k$, then $a_{8k} = \sin(2k\pi) = 0$. This subsequence converges to 0.
   - Pick $n_k = 8k + 2$, then $a_{8k+2} = \sin(2k\pi + \frac{\pi}{2}) = 1$. This subsequence converges to 1.
   - Because we found two subsequences with different limits (0 and 1), the sequence diverges.

   Answer: Diverges
