e exists: (1+1/n)^n converges

Lesson 27 · Module 2 — Sequences · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Using the binomial expansion $a_n = \sum_{k=0}^{n} \frac{1}{k!}(1 - \frac{1}{n}) \dots (1 - \frac{k-1}{n})$, prove that for any $n \geq 2$, the term for $k = 2$ is less than or equal to $\frac{1}{2}$.
2. Show that the sequence $a_n = (1 + \frac{1}{n})^n$ is strictly increasing for $n \geq 1$ by comparing the terms of the binomial expansion for a_n and a_{n+1}.
3. Prove that $a_n < 3$ for all n by using the bound $k! \geq 2^{k-1}$ for $k \geq 1$.

Solutions

1. Using the binomial expansion $a_n = \sum_{k=0}^{n} \frac{1}{k!}(1 - \frac{1}{n}) \dots (1 - \frac{k-1}{n})$, prove that for any $n \geq 2$, the term for $k = 2$ is less than or equal to $\frac{1}{2}$.

   - The term for $k = 2$ is $\frac{1}{2!}(1 - \frac{1}{n})(1 - \frac{2}{n})$
   - Since $n \geq 2$, the factors $(1 - \frac{1}{n})$ and $(1 - \frac{2}{n})$ are both ≤ 1
   - Therefore, the product is $\leq \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

   Answer: \frac12!(1 - \frac1n)(1 - \frac2n) \leq \frac12

2. Show that the sequence $a_n = (1 + \frac{1}{n})^n$ is strictly increasing for $n \geq 1$ by comparing the terms of the binomial expansion for a_n and a_{n+1}.

   - Write a_n as a sum of terms $T_{k,n} = \frac{1}{k!} \prod_{j=1}^{k-1}(1 - \frac{j}{n})$
   - Observe that $1 - \frac{j}{n} < 1 - \frac{j}{n+1}$ for all $j \geq 1$
   - Thus $T_{k,n} < T_{k,n+1}$ for all $k \leq n$
   - Since a_{n+1} has all the increased terms of a_n plus an additional positive term $T_{n+1,n+1}$, $a_{n+1} > a_n$

   Answer: a_n+1 ¿ a_n

3. Prove that $a_n < 3$ for all n by using the bound $k! \geq 2^{k-1}$ for $k \geq 1$.

   - Start with $a_n \leq \sum_{k=0}^{n} \frac{1}{k!}$
   - Separate the first term: $a_n \leq 1 + \sum_{k=1}^{n} \frac{1}{k!}$
   - Substitute the bound $k! \geq 2^{k-1}$: $a_n \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}}$
   - The sum is a geometric series: $\sum_{i=0}^{n-1} (1/2)^i < \frac{1}{1-1/2} = 2$
   - Thus $a_n < 1 + 2 = 3$

   Answer: a_n ¡ 3
